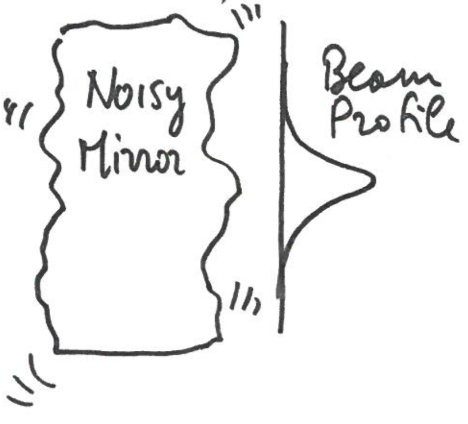
Thermal Noise in ~~a~~ GW Detector

Let's consider only the observable represented by the displacement of the reflecting surface of a mirror along the direction of the laser beam.
We will not consider any other source of phase fluctuations on the reflected light beam.
First of all, since the frequency f is always be less than 10^4 Hz, the F-D Theorem is considered in its classical limit even at cryogenic temperatures:

$$S_{xx}(\omega) = \frac{4kT}{\omega}\alpha'' \quad (1) \quad \text{for } T > 10^{-5}\,K$$

Second, we need to identify our observable.



If we call with $w(x,y,t)$ the shape of the front surface at time t, our observable $x(t)$ will be:

$$x(t) = \int_S w(x,y,t)\cdot B(x,y)\,dx\,dy \quad (2)$$

where $B(x,y)$ is the function giving the beam profile; usually it is a Gaussian.
B is normalized to unity: $\int_S B(x,y)\,dx\,dy = 1 \quad (3)$
which means $[B] = m^{-2}$. $\longrightarrow$ $[w] = m$

Third, we have to find the force that drives $X(t)$.
If on the surface element $dxdy$ we apply a pressure $P(x,y)$ we create a force $dF = P(x,y)\,dxdy$.
Therefore, a general interacting term in the Hamiltonian of the mirror is indeed

$$V_I = -\int_S W \cdot P \cdot dxdy \qquad (4)$$

Now, if $P(x,y) = F_0(t) \cdot B(x,y)$, eq.(4) gives:

$$V_I = -F_0(t) \cdot \int_S W \cdot B\, dxdy \overset{(2)}{=} -F_0(t) \cdot X(t) \qquad (5)$$

Before going further there is a detail that we need to fix. We have supposed that the mirror is free. The integral of the pressure P gives the total force:

$$F_T(t) = \int_S P\, dS = F_0(t) \cdot \int_S B\, dS = F_0(t) \qquad (6)$$

Since we have to use the pressure P to calculate the response α of the system, when $\omega \to 0$ the total force is $\neq 0$ and the mirror start to move as a rigid body. The resulting motion is that of a pendulum, therefore people liked to separate the observable $X(t)$ in two components: the pendulum and the mirror deformation components:

$$X(t) = X^{(p)}(t) + X^{(m)}(t) \qquad (7)$$

For the moment we suppose the mass perfectly free, hence the dynamic of $x^{(p)}$ is noiseless (and non-dissipative).

Equation (5) has clarified one point: the system needs to be sollicitated by a pressure of the form:

$$Re\left\{F_0 e^{i\omega t} \cdot B(x,y)\right\} \quad (8)$$

As a result of this excitation a deformation $w(x,y,t)$ is produced on the surface, and on the whole body of the mirror $w(x,y,z,t)$.

At this point Y. Levin suggested a clever way to calculate α''. Following what we have done in the chapter dedicated to the F-D theorem, eq.(16) there reads:

$$\boxed{W_{diss.} = \pi f_0^2 \alpha''} \quad (9)$$

The energy dissipeded in 1 cycle, that we derived from the work done by the external force, can be calculated as energy dissipaded INSIDE the system during the action of the external force. The algorithm of the thermal noise calculation is as follows:

$$Re\{f_0 e^{i\omega t} \cdot B(x,y)\} \longrightarrow w(x,y,z,t) \longrightarrow W_{diss} \longrightarrow$$

$$\longrightarrow \alpha'' = \frac{W_{diss}}{\pi f_0^2} \longrightarrow S_{xx}(\omega) = \frac{4kT}{\omega} \frac{W_{diss}}{\pi f_0^2} \quad (10)$$

Where the energy is dissipated inside a material?
Which are the mechanisms?

Limiting our analysis to frequency below few tens of MHz, the main ~~mechan~~ dissipation of energy comes from the ANELASTICITY.

ANELASTICITY = RETARDED ELASTICITY

Using the σ-~~stress~~ notation:

$\sigma_1 = \sigma_{xx}$ $\sigma_2 = \sigma_{yy}$ $\sigma_3 = \sigma_{zz}$ $\sigma_4 = \sigma_{yz}$ $\sigma_5 = \sigma_{xz}$ $\sigma_6 = \sigma_{xy}$

The standard Hooke law:

$$\varepsilon_i = S_{ij}\,\sigma_j \qquad (11)$$

S is the compliance tensor

needs to be replaced by:

$$\varepsilon_i(t) = \int_{-\infty}^{t} S_{ij}(t-s)\,\sigma_j(s)\,ds \qquad (12)$$

The delayed response produces imaginary elastic coefficients in the frequency domain. From (12):

$$\varepsilon_i(\omega) = \left[S'_{ij}(\omega) + i\,S''_{ij}(\omega)\right]\sigma_j(\omega) \qquad (13)$$

and their inverse:

$$\sigma_i(\omega) = \left[C'_{ij}(\omega) + i\,C''_{ij}(\omega)\right]\varepsilon_j(\omega) \qquad (14)$$

C is the stiffness tensor [I KNOW: IT IS CRAZY!]

If we consider a volume dV so small that the stress and strain are considered constant therein, the internal energy of that volume can change with the work (infinitesimal)

$$\sigma_i \, d\varepsilon_i \, dV \qquad (15)$$

One can use the (14) to replace σ_i and express the energy (15) with the deformation only, deformation that comes from the action of the pressure (8).

Let's calculate the work in a cycle (because we need to work out the W_{diss}).

$$(15) = Re\{[C'_{ij} + i C''_{ij}] \varepsilon^o_j e^{i\omega t}\} \cdot Re\{i\omega \varepsilon^o_i e^{i\omega t}\} dt \, dV =$$

$$= Re\{|C_{ij}| e^{-i\varphi_{ij}} \varepsilon^o_j e^{i\omega t}\} \cdot Re\{i\omega \varepsilon^o_i e^{i\omega t}\} dt \, dV =$$

$$= \frac{1}{4} |C_{ij}| \varepsilon^o_j \varepsilon^o_i \{e^{i(-\varphi_{ij}+\omega t)} + e^{-i(-\varphi_{ij}+\omega t)}\}\{i\omega e^{i\omega t} - i\omega e^{-i\omega t}\} dt \, dV$$

$$= \frac{1}{4} |C_{ij}| \varepsilon^o_j \varepsilon^o_i \{e^{-i\varphi_{ij}}(-i\omega) + i\omega e^{+i\varphi_{ij}} + \dots\} dt \, dV =$$

$$\cong -\frac{1}{4} |C_{ij}| \varepsilon^o_j \varepsilon^o_i \{\omega\varphi_{ij} + \omega\varphi_{ij}\} dt \, dV = -\frac{|C_{ij}| \varepsilon^o_j \varepsilon^o_i}{2} \varphi_{ij} \, \omega \, dt \, dV \qquad (16)$$

We used $\varphi_{ij} << 1$ and neglect the time dependent terms.

Integrating in time from 0 to T:

$$\int_0^T (15) \, dt = -2\pi \frac{|C_{ij}|}{2} \varepsilon^o_j \varepsilon^o_i \, \varphi_{ij} \, dV = -2\pi \, E_{stored} \cdot \varphi_{ij} \, dV \qquad (17)$$

So the energy dissipated in 1 cycle is

$$W_{diss} = \int 2\pi E_{stored} \varphi_{ij} \, dV = 2\pi \frac{1}{2} \int |C_{ij}| \varepsilon^o_j \varepsilon^o_i \varphi_{ij} \, dV \qquad (18)$$