THE TT FRAME

In order to simplyfy the linearized Eistein's equation we have chosen a specific gauge of 4 functions $\chi^\mu(x)$ in order to force $h^\mu{}_\mu = h = 0$ and $h^{0i} = 0$ when the coordinate change as $x'^\mu = x^\mu + \chi^\mu(x)$.

Now, we are an observer that uses a free falling frame and we decide to use the TT gauge. The free falling frame assures us that locally the metric is flat, then a GW is passing by and we choose to change coordinates.

Still we have

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

but in the TT gauge $h_{\mu\nu} \to h^{TT}_{\mu\nu}$: $h^{TT} = 0$ and $h^{TT}_{0i} = 0$

From the latter we derive that $h^{TT}_{00} = 0$ too.

What happens to a mass initially at rest?

$$\left.\frac{dx^i}{d\tau}\right|_{\tau=0} = 0 \qquad (1)$$

the geodesic equation is:

$$\frac{du^i}{d\tau} + \Gamma^i_{\mu\nu} u^\mu u^\nu = 0 \qquad (2)$$

Considering (1): $\left.\frac{du^i}{d\tau}\right|_{\tau=0} + \Gamma^i_{00} u^0 u^0 = 0$ (3)

From the definition of the linearized Γ:

$\Gamma^\mu_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}(\partial_\nu h_{\rho\sigma} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho})$ we have

$$\Gamma^i_{00} = \frac{1}{2}\eta^{i\sigma}(\partial_0 h_{0\sigma} + \partial_0 h_{0\sigma} - \partial_\sigma h_{00}) =$$

$$= \partial_0 h_0{}^i - \frac{1}{2}\partial^i h_{00}$$

Since in the TT gauge $h_{0\mu} = 0$ at all points x^ν then their derivatives are also zero, hence $\Gamma^i_{00} = 0$, hence:

$$\left.\frac{du^i}{d\tau}\right|_{\tau=0} = 0 \quad (4)$$

The mass does not accelerate and since $u^i = 0$ its position remains at rest.

Now, if at time $\tau = 0$ two masses are separated by ξ^i and $u^i_{(1)} = u^i_{(2)} = 0$ (both at rest at $\tau = 0$) let's see what happens to the separation ξ^μ.

Using the geodesic deviation equation:

$$\frac{d^2\xi^i}{d\tau^2} + \xi^j\left(\partial_j \Gamma^i_{00}\right)\left(\frac{dX^0}{d\tau}\right)^2 + 2\Gamma^i_{0k}\frac{dX^0}{d\tau}\underbrace{\frac{d\xi^k}{d\tau}}_{0} = 0$$

$$\Gamma^i_{00} = 0 \rightarrow \partial_j \Gamma^i_{00} = \partial_j \partial_0 h_0{}^i - \frac{1}{2}\partial_j \partial^i h_{00} = 0$$

because $\Gamma^i_{00} = 0$ everywhere. Hence:

$$\left.\frac{d^2\xi^i}{d\tau^2}\right|_{\tau=0} = 0 \qquad (5)$$

The message is that in the TT frame the coordinates stretch themselves accordingly to the passing GW in order to keep the coordinates of rest points at rest.

If two rest points are at distance $\vec{L}$ then:

$$s^2 = \int ds^2 = \int (\eta_{ij} + h^{TT}_{ij})\,dx^i dx^j = L^2 + h^{TT}_{ij} L^i L^j$$

if $|\vec{L}|$ is so small that h^{TT}_{ij} does not change signif.

The proper distance s is then

$$s \simeq L + \frac{1}{2} h^{TT}_{ij} \underbrace{\frac{L^i L^j}{L}}_{\hat{n}^i} \qquad (6) \rightarrow \ddot{s}_j \cong \frac{1}{2}\ddot{h}^{TT}_{ij} s^i$$