THE QUADRUPOLE MASS RADIATION

$$h^{TT}_{ij}(t,\vec{x}) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}\int d^3x'\, x'^k x'^l\, \ddot{T}^{00}\left(t-\frac{r}{c},\vec{x}'\right) \quad (1)$$

$$= \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}\,\ddot{M}^{kl}\left(t-\frac{r}{c}\right) \quad (2)$$

The moment M^{kl} can be composed by a traceless component + a matrix proportional to the identity:

$$M^{kl} = Q^{kl} + \frac{M^{ii}}{3}\delta^{kl} \quad (3) \quad \text{where}$$

$$Q^{kl} = M^{kl} - \frac{M^{ii}}{3}\delta^{kl} \quad (4)$$

is known from mechanics as quadrupole mass moment.

$$Q^{kl}(t) = \int d^3x'\,\rho(t,\vec{x}')\left(x^k x^l - \frac{1}{3}x^2\delta^{kl}\right) \quad (5)$$

(2) becomes:

$$h^{TT}_{ij}(t,\vec{x}) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}\,\ddot{Q}^{kl}\left(t-\frac{r}{c}\right) \quad (6)$$

because Λ applied to the δ give 0.

Let's see how Λ operates:

For an observer situated far away along z $(\hat{n}=\hat{k})$ the projection operator is:

$$P_{ij}(\hat{n}) = \delta_{ij} - n_i n_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (7)$$ then

$$\Lambda_{ij,kl} Q^{kl} = \underbrace{P_{ik} P_{jl} Q^{kl}}_{\tilde{P}^T \tilde{Q} \tilde{P}} - \frac{1}{2} P_{ij} \underbrace{P_{kl} Q^{kl}}_{\tilde{P}\cdot \mathrm{Tr}(\tilde{P}\tilde{Q})} \quad (8)$$

$$\tilde{Q}\tilde{P} = \begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a & d & 0 \\ d & b & 0 \\ f & e & 0 \end{pmatrix}$$

$$\tilde{P}^T\tilde{Q}\tilde{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & d & 0 \\ d & b & 0 \\ f & e & 0 \end{pmatrix} = \begin{pmatrix} a & d & 0 \\ d & b & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (9)$$

$$\tilde{P}\tilde{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix} = \begin{pmatrix} a & d & f \\ d & b & e \\ 0 & 0 & 0 \end{pmatrix} \quad \mathrm{Tr}(\tilde{P}\tilde{Q}) = a+b \quad (10)$$

Then

$$\tilde{\Lambda}\tilde{Q} = \begin{pmatrix} a & d & 0 \\ d & b & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} (a+b) = \begin{pmatrix} \frac{a-b}{2} & d & 0 \\ d & \frac{b-a}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (11)$$

From (6)

$$h_{ij}^{TT}(t,z) = \frac{1}{z}\frac{2G}{c^4}\begin{pmatrix} \frac{\ddot{Q}_{11}-\ddot{Q}_{22}}{2} & \ddot{Q}_{12} & 0 \\ \ddot{Q}_{12} & \frac{\ddot{Q}_{22}-\ddot{Q}_{11}}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{(t-\frac{r}{c})} \quad (12)$$

hence

$$h_+ = \frac{1}{z}\frac{2G}{c^4}\frac{\ddot{Q}_{11}-\ddot{Q}_{22}}{2} \quad (13)$$

$$h_\times = \frac{1}{z}\frac{2G}{c^4}\ddot{Q}_{12} \quad (14)$$