$$\mathcal{L} = \frac{1}{2}\mu \dot{\vec{r}}^{\,2} + G\frac{M\mu}{r} = \frac{1}{2}\mu\left(r^2\dot{\theta}^2 + \dot{r}^2\right) + G\frac{M\mu}{r}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q} \qquad \begin{cases} \mu r^2 \ddot{\theta} = 0 \\ \mu \ddot{r} = \mu r \dot{\theta}^2 - \dfrac{GM\mu}{r^2} \end{cases}$$

If we suppose $\ddot{r} = 0$ then $\dot{\theta}^2 = \frac{GM}{r^3}$ hence the kinetic energy is

$$K = \frac{1}{2}\mu\left(r^2\dot{\theta}^2 + \dot{r}^2\right) = \frac{1}{2}\mu\frac{GM}{r} = \frac{U}{2} \qquad \boxed{K = \frac{U}{2}}$$

$$K = \frac{1}{2}\mu\left(r\cdot\dot{\theta}\right)^2 = \frac{1}{2}\mu v^2 = \frac{GM\mu}{2r} \to \frac{v^2}{c^2} = \frac{GM}{c^2 r}$$

$$\frac{v^2}{c^2} = \frac{R_S}{2r} \quad \text{where } R_S = 2\frac{GM}{c^2} \quad \text{Schwarzschild Radius}$$

$$[G] = \left[\frac{F\cdot l^2}{m^2}\right] = \left[\frac{m\, l\, l^2}{m^2 t^2}\right] = \left[\frac{l\cdot v^2}{m}\right]$$

$$1 \gg \frac{v}{c} \sim \frac{\omega\cdot d}{c} = \frac{2\pi\cdot d}{T\cdot c} = \frac{2\pi d}{\lambda} = \frac{d}{\bar{\lambda}} \to d \ll \bar{\lambda}$$

If $f = 100\,Hz \to T = 10^{-2}\,s$ $\quad \bar{\lambda} = \frac{T\cdot c}{2\pi} = \frac{10^{-2}\cdot 3\cdot 10^{8}}{2\cdot\pi} \simeq 5\cdot 10^{5}\,m = 500\,km$

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \quad (1)$$

The Green's function:

$$\Box_x G(x-x') = \delta^4(x-x') \quad (2)$$

since $T_{\mu\nu}(x) = \int d^4x' \, \delta^4(x-x') \, T_{\mu\nu}(x')$

if $\bar{h}_{\mu\nu}(x) = \int d^4x' \, G(x-x') \left[-\frac{16\pi G}{c^4} T_{\mu\nu}(x') \right]$ (3) then:

$$\Box_x \bar{h}_{\mu\nu}(x) = \int d^4x' \, \delta^4(x-x') \left[\quad \right] = -\frac{16\pi G}{c^4} T_{\mu\nu}(x)$$

From (2) the Green's function is:

$$G(x-x') = -\frac{1}{4\pi |\vec{x}-\vec{x}'|} \delta(x^0_{ret} - x'^0) \quad (4)$$

where $t_{ret} = \frac{x^0_{ret}}{c} = t - \frac{|\vec{x}-\vec{x}'|}{c}$

The solution is then:

(3)':

$$\bar{h}_{\mu\nu}(x) = \frac{4G}{c^4} \int d^3x' \frac{1}{|\vec{x}-\vec{x}'|} T_{\mu\nu}\left(\overset{t}{\overset{\downarrow}{t}} - \frac{|\vec{x}-\vec{x}'|}{c}, \vec{x}' \right) \quad (5)$$

$\hookrightarrow (t, \vec{x})$

As we have seen it is always possible to find 4 $\chi(x)$ that will make $\bar{h}_{0\nu}=0 \quad \forall \nu$. Then, in order to find the TT solution, we have to apply the operator Λ. With that, since the trace is 0, $\bar{h}^{TT}=h^{TT}$.

$$\bar{h}_{\mu\nu} \rightarrow \bar{h}_{ij} \rightarrow h^{TT}_{ij} = \Lambda_{ij,km}\,\bar{h}_{km} \qquad \begin{matrix}\mu,\nu=0,\dots,3\\ i,j,k,m=1,\dots,3\end{matrix}$$

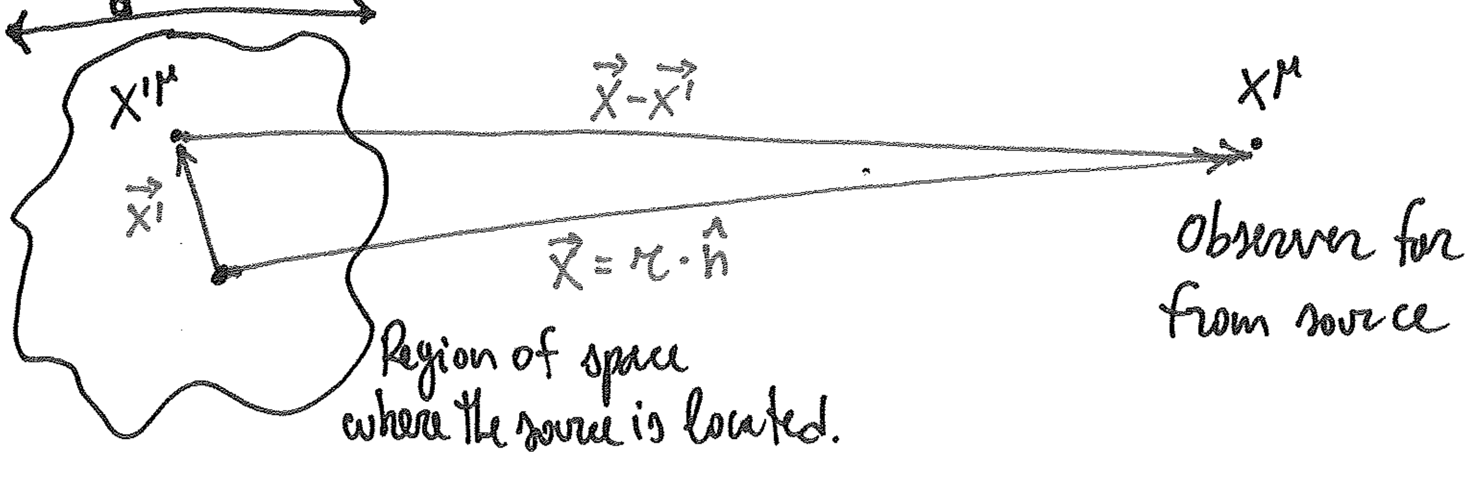
In order to evaluate (5) M. Maggiore suggests to write T_{kl} in the Fourier space.

$$T_{kl}\left(t-\frac{|\vec{x}-\vec{x}'|}{c},\vec{x}'\right)=\int\frac{d^4k}{(2\pi)^4}\,\tilde{T}_{kl}(\omega,\vec{k})\,e^{+ik_\mu x'^\mu} \qquad (6)$$

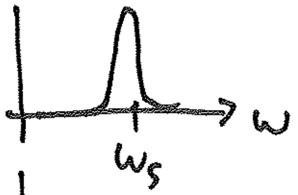
where $k^\mu=\left(\frac{\omega}{c},\vec{k}\right) \longrightarrow k_\mu=\left(-\frac{\omega}{c},\vec{k}\right)$ (7) and

$$ik_\mu x'^\mu=-i\frac{\omega}{c}\left(c\cdot t-|\vec{x}-\vec{x}'|\right)+i\vec{k}\cdot\vec{x}'$$

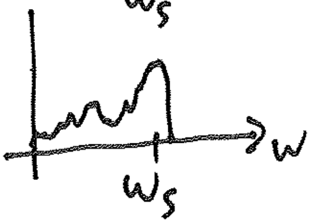
$\left[x'^\mu=(ct',\vec{x}') \text{ but with (4) } ct'=ct-|\vec{x}-\vec{x}'|\right]$



Non-relativistic sources $\equiv v \ll c \rightarrow \frac{d}{ƛ} \ll 1$



If $\omega_g = 2\pi \cdot 100 \rightarrow ƛ = 500$ km



For BH binaries $\boxed{d \lesssim 500\,\text{km}}$

From the drawing:

$$|\vec{x} - \vec{x}'| = r - \vec{x}' \cdot \hat{n} \rightarrow \omega \frac{|\vec{x}-\vec{x}'|}{c} = \frac{\omega}{c} \cdot r - \frac{\omega}{c} \vec{x}' \cdot \hat{n} =$$

$$= \frac{\omega}{c} \cdot r - \frac{\vec{x}' \cdot \hat{n}}{ƛ} \qquad (8)$$

and $\frac{\vec{x}' \cdot \hat{n}}{ƛ} < \frac{d}{ƛ} \ll 1$

therefore, one can write

$$e^{-i\omega\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)} = e^{-i\left(\omega t - \frac{\omega}{c} r + \frac{\omega}{c} \vec{x}' \cdot \hat{n}\right)} =$$

$$\simeq e^{-i\left(\omega t - \frac{\omega}{c} r\right)} \left[1 - i \frac{\omega}{c} \vec{x}' \cdot \hat{n} - \frac{1}{2}\left(\frac{\omega}{c} \vec{x}' \cdot \hat{n}\right)^2 + \dots\right]$$

$$= e^{-i\left(\omega t - \frac{\omega}{c} r\right)} \left[1 + \vec{x}' \cdot \hat{n}\left(-i\frac{\omega}{c}\right) + \frac{1}{2}(\vec{x}' \cdot \hat{n})^2 \left(-i\frac{\omega}{c}\right)^2 + \dots\right]$$

The energy-momentum tensor (6) becomes:

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$$T_{kl}\left(t-\frac{r}{c}+\frac{\vec{x}'\cdot\hat{n}}{c},\vec{x}'\right)=\int\frac{d^4k}{(2\pi)^4}\tilde{T}_{kl}(\omega,\vec{k})\,e^{-i\left[\omega t-\frac{\omega r}{c}-\vec{k}\cdot\vec{x}'\right]}+$$

$$+\vec{x}'\cdot\hat{n}\int\frac{d^4k}{(2\pi)^4}\tilde{T}_{kl}(\omega,\vec{k})\,e^{-i[\ ]}\cdot\left(-i\frac{\omega}{c}\right)+$$

$$+\frac{1}{2}(\vec{x}'\cdot\hat{n})^2\int\frac{d^4k}{(2\pi)^4}\tilde{T}_{kl}(\omega,\vec{k})\,e^{-i[\ ]}\cdot\left(-i\frac{\omega}{c}\right)^2+\ldots=$$

$$\boxed{\begin{array}{l}\vec{x}'\cdot\hat{n}=n_m x'^m\\ \text{Since } m=1,2,3\\ \text{then } n_m=n^m\end{array}}$$

$$=T_{kl}\left(t-\frac{r}{c},\vec{x}'\right)+$$

$$+\frac{\vec{x}'\cdot\hat{n}}{c}\,\partial_t T_{kl}\left(t-\frac{r}{c},\vec{x}'\right)+$$

$$+\frac{1}{2}\left(\frac{\vec{x}'\cdot\hat{n}}{c}\right)^2\partial_t^2 T_{kl}\left(t-\frac{r}{c},\vec{x}'\right)+\ldots \qquad (9)$$

From (5) the TT solution is:

$$h_{\mu\nu}^{TT}(t,\vec{x})\simeq\frac{4G}{c^4}\frac{1}{r}\Lambda_{\mu\nu,kl}\Big[\int d^3x'\,T_{kl}\left(t-\frac{r}{c},\vec{x}'\right)+$$

$$+\frac{n_m}{c}\int d^3x'\,x'^m\,\dot{T}_{kl}\left(t-\frac{r}{c},\vec{x}'\right)+$$

$$+\frac{1}{2}\frac{n_m\,n_j}{c^2}\int d^3x'\,x'^m x'^j\,\ddot{T}_{kl}\left(t-\frac{r}{c},\vec{x}'\right)$$

$$+\ldots \qquad (10)$$

(10) can be written as :

$$h^{TT}_{\mu\nu}(t,\vec{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{\mu\nu,kl}\Big[S_{kl}(t-\tfrac{r}{c}) + \frac{n_m}{c}\dot{S}^{m}_{kl}(t-\tfrac{r}{c}) + \frac{1}{2}\frac{n_m \cdot m_j}{c^2}\ddot{S}^{mj}_{kl}(t-\tfrac{r}{c}) + \dots\Big] \quad (11)$$

where
$$S_{kl}(t-\tfrac{r}{c}) = \int d^3x' \, T_{kl}(t-\tfrac{r}{c},\vec{x}') \quad (12)$$

$$S^{m}_{kl}(t-\tfrac{r}{c}) = \int d^3x' \cdot x'^m \cdot T_{kl}(t-\tfrac{r}{c},\vec{x}') \quad (13)$$

$$S^{mj}_{kl}(t-\tfrac{r}{c}) = \int d^3x' \, x'^m \cdot x'^j \, T_{kl}(t-\tfrac{r}{c},\vec{x}') \quad (14)$$

(11) is the multi mode expansion of GW valid for non relativistic sources.

The leading term is the first, proportional to S_{kl}.

<u>The TRICK :</u>

$$S_{kl} = \frac{1}{2}(S_{kl} + S_{lk}) = \frac{1}{2}(\dot{P}_{k,l} + \dot{P}_{l,k}) = \frac{1}{2}\ddot{M}_{kl}$$ where

$$M_{kl} = \int d^3x' \, x'_k x'_l \, T^{00}$$

THE QUADRUPOLE RADIATION

$$h^{TT}_{ij}(t,\vec{x}) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}\int d^3x' \, x'^k x'^l \, \ddot{T}^{00}(t-\tfrac{r}{c},\vec{x}')$$