$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2}\eta^{\gamma\delta}\left(\partial_\alpha h_{\delta\beta} + \partial_\beta h_{\alpha\delta} - \partial_\delta h_{\alpha\beta}\right)$$

$$R^{\alpha}_{\mu\beta\nu} \simeq \partial_\beta \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} =$$

$$= \frac{1}{2}\eta^{\alpha\delta}\left(\partial_\beta\partial_\mu h_{\delta\nu} + \partial_\beta\partial_\nu h_{\mu\delta} - \partial_\beta\partial_\delta h_{\mu\nu}\right) - \frac{1}{2}\eta^{\alpha\delta}\left(\partial_\nu\partial_\mu h_{\delta\beta} + \partial_\nu\partial_\beta h_{\mu\delta} - \partial_\nu\partial_\delta h_{\mu\beta}\right)$$

$$=$$

$$= \frac{1}{2}\left(\partial_\beta\partial_\mu h^{\alpha}{}_{\nu} + \cancel{\partial_\beta\partial_\nu h^{\alpha}{}_{\mu}} - \partial_\beta\partial^\alpha h_{\mu\nu} - \partial_\nu\partial_\mu h^{\alpha}{}_{\beta} + \partial_\nu\partial^\alpha h_{\mu\beta}\right)$$

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} =$$

$$= \frac{1}{2}\left(\partial_\alpha\partial_\mu h^{\alpha}{}_{\nu} - \partial_\alpha\partial^\alpha h_{\mu\nu} - \partial_\mu\partial_\nu h^{\alpha}{}_{\alpha} + \partial_\nu\partial^\alpha h_{\mu\alpha}\right)$$

$$\Box = -\frac{1}{c^2}\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2 \qquad -h_{00} + h_{11} + h_{22} + h_{33} = h$$

$$=$$

$$R = \eta^{\mu\nu}R_{\mu\nu} = \frac{1}{2}\left(\partial_\alpha\partial_\mu h^{\alpha\mu} - \Box h_{\mu}{}^{\mu} - \Box h^{\alpha}{}_{\alpha} + \partial^\mu\partial^\alpha h_{\mu\alpha}\right) =$$

$$=$$

$$= \partial^\alpha\partial^\beta h_{\alpha\beta} - \Box h$$

$$\frac{1}{2}\left(\partial_\mu\partial_\alpha h^\alpha{}_\nu + \partial_\nu\partial_\alpha h^\alpha{}_\mu - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right) + \frac{1}{2}\eta_{\mu\nu}\left(\partial^\alpha\partial^\beta h_{\alpha\beta} - \Box h\right) =$$
$$= \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$\bar h_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

$$\bar h = \eta^{\mu\nu}\bar h_{\mu\nu} = h - \frac{1}{2}\eta^{\mu\nu}\eta_{\mu\nu} h = h - \frac{1}{2} 4 h = -h$$ hence:

$$h_{\mu\nu} = \bar h_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu} h = \bar h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar h$$

$$2G_{\mu\nu} = \partial_\mu\partial_\alpha \bar h^\alpha{}_\nu + \partial_\nu\partial_\alpha \bar h^\alpha{}_\mu - \Box \bar h_{\mu\nu} + \partial_\mu\partial_\nu \bar h - \frac{1}{2}\eta_{\mu\nu}\left(\partial^\alpha\partial^\beta \bar h_{\alpha\beta} - \Box \bar h\right)$$
$$- \frac{1}{2}\left(\partial_\mu\partial_\alpha \eta^\alpha{}_\nu + \partial_\nu\partial_\alpha \eta^\alpha{}_\mu - \Box \eta_{\mu\nu}\right)\bar h + \frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial^\beta\eta_{\alpha\beta}\bar h =$$

$$= \partial_\mu\partial_\alpha \bar h^\alpha{}_\nu + \partial_\nu\partial_\alpha \bar h^\alpha{}_\mu - \Box \bar h_{\mu\nu} + \cancel{\partial_\mu\partial_\nu \bar h} + \frac{1}{2}\eta_{\mu\nu}\left(\partial^\alpha\partial^\beta \bar h_{\alpha\beta} + \Box \bar h\right)$$
$$- \frac{1}{2}\left(\cancel{\partial_\mu\partial_\nu} + \cancel{\partial_\nu\partial_\mu} - \eta_{\mu\nu}\Box\right)\bar h + \frac{1}{2}\eta_{\mu\nu}\Box \bar h =$$

$$2G_{\mu\nu} = \partial_\mu \partial^\alpha h_{\alpha\nu} + \partial_\nu \partial^\alpha h_{\alpha\mu} - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h^\alpha_\alpha +$$

$$- \eta_{\mu\nu} \left(\partial^\alpha \partial^\beta h_{\alpha\beta} - \Box h^\alpha_\alpha \right) \qquad h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \bar{h}$$

$$2G_{\mu\nu} = \partial_\mu \partial^\alpha \bar{h}_{\alpha\nu} + \partial_\nu \partial^\alpha \bar{h}_{\alpha\mu} - \Box \bar{h}_{\mu\nu} - \partial_\mu \partial_\nu \bar{h}^\alpha_\alpha +$$

$$- \eta_{\mu\nu} \left(\partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} - \Box \bar{h}^\alpha_\alpha \right) +$$

$$- \frac{1}{2} \Big\{ \left(\partial_\mu \partial_\nu + \partial_\nu \partial_\mu - \eta_{\mu\nu} \Box - \eta^\alpha_\alpha \partial_\mu \partial_\nu \right) \bar{h} +$$

$$- \eta_{\mu\nu} \left(\Box - \eta^\alpha_\alpha \Box \right) \bar{h} \Big\} =$$

$$\bar{h}^\alpha_\alpha = \bar{h}$$

$$\eta^\alpha_\alpha = 4$$

$$= \partial_\mu \partial^\alpha \bar{h}_{\alpha\nu} + \partial_\nu \partial^\alpha \bar{h}_{\alpha\mu} - \Box \bar{h}_{\mu\nu} - \cancel{\partial_\mu \partial_\nu \bar{h}} +$$

$$- \eta_{\mu\nu} \left(\partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} - \cancel{\Box \bar{h}} \right) +$$

$$- \cancel{\partial_\mu \partial_\nu \bar{h}} + \frac{1}{2} \cancel{\eta_{\mu\nu} \Box \bar{h}} + 2 \cancel{\partial_\mu \partial_\nu \bar{h}} - \cancel{\frac{3}{2} \eta_{\mu\nu} \Box \bar{h}} =$$

$$= \partial_\mu \partial^\alpha \bar{h}_{\alpha\nu} + \partial_\nu \partial^\alpha \bar{h}_{\alpha\mu} - \Box \bar{h}_{\mu\nu} - \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta}$$

4

If $\partial^\alpha \bar{h}_{\alpha\beta} = 0 \ \forall \beta \rightarrow \quad \Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$

$x'^\mu = x^\mu + \xi^\mu(x) \qquad ds^2 = ds'^2$

$$g_{\mu\nu} dx^\mu dx^\nu = g'_{\mu\nu} dx'^\mu dx'^\nu$$

$$g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\alpha} dx'^\alpha \cdot \frac{\partial x^\nu}{\partial x'^\beta} dx'^\beta \rightarrow g'_{\alpha\beta} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\alpha} \cdot \frac{\partial x^\nu}{\partial x'^\beta}$$

$x^\mu = x'^\mu - \xi^\mu(x') \qquad \frac{\partial x^\mu}{\partial x'^\alpha} = \delta^\mu_\alpha - \partial_\alpha \xi^\mu = \eta^\mu{}_\alpha - \partial_\alpha \xi^\mu$

$$g'_{\alpha\beta} = g_{\mu\nu}\left(\eta^\mu{}_\alpha - \partial_\alpha \xi^\mu\right)\left(\eta^\nu{}_\beta - \partial_\beta \xi^\nu\right) = \qquad \boxed{\partial \xi = O(h)}$$

$$\simeq g_{\alpha\beta} - g_{\alpha\nu} \partial_\beta \xi^\nu - g_{\mu\beta} \partial_\alpha \xi^\mu =$$

$$\simeq g_{\alpha\beta} - \partial_\beta \xi_\alpha - \partial_\alpha \xi_\beta = \eta_{\alpha\beta} + \underbrace{h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha}_{h'_{\alpha\beta}}$$

$$h_{\alpha\beta} \rightarrow h'_{\alpha\beta} \rightarrow \bar{h}'_{\alpha\beta} = h'_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h'^\mu{}_\mu = h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha$$
$$-\frac{1}{2}\eta_{\alpha\beta} h + \frac{1}{2}\eta_{\alpha\beta}(\partial^\alpha \xi_\alpha + \partial^\beta \xi_\beta)$$

$$= \bar{h}_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha + \eta_{\alpha\beta} \partial^\mu \xi_\mu$$

$$\partial^\alpha \bar{h}'_{\alpha\beta} = \partial^\alpha \bar{h}_{\alpha\beta} - \Box \xi_\beta - \cancel{\partial_\beta \partial^\alpha \xi_\alpha} + \cancel{\partial_\beta \partial^\mu \xi_\mu} = 0 \qquad \boxed{\Box \xi_\beta = \partial^\alpha \bar{h}_{\alpha\beta}}$$

Given a $h'_{\mu\nu}$ $\exists\, \xi_\mu : \partial^\mu \bar{h}_{\mu\nu} = 0 \Rightarrow \Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$

$h_{\mu\nu} = h'_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$ and $\Box \xi_\beta = \partial^\alpha \bar{h}'_{\alpha\beta}$

$h_{\mu\nu}$ has 6 independent components: $10 - 4(\partial^\mu h_{\mu\nu} = 0)$

One can choose another set of 4 functions $\chi^\mu : \Box \chi^\mu = 0$ to change again the coordinates $x^\mu \to x^\mu + \xi^\mu \to x^\mu + \xi^\mu + \chi^\mu$

Since $\Box \chi^\mu = 0$ the new $\bar{h}_{\mu\nu}$ will have the same gauge $\partial^\mu h_{\mu\nu} = 0$

$$h''_{\mu\nu} = h'_{\mu\nu} - \partial_\mu \chi_\nu - \partial_\nu \chi_\mu$$

We choose χ^μ so that:

$$h''^{\alpha}_{\ \alpha} = \eta^{\mu\nu} h''_{\mu\nu} = h' - 2\partial_\mu \chi^\mu = 0$$ ← We use χ^0 to satisfy this equation

if $h''^{\alpha}_{\ \alpha} = 0$ then $\boxed{\bar{h}''_{\mu\nu} = h''_{\mu\nu}}$ $h''_{\mu\nu}$ is TRACELESS

The other 3 χ^μ are used to satisfy $h''^{0i} = 0 \quad i = 1,2,3$

$$\left.\begin{array}{l} \partial^\mu h_{\mu\nu} = 0 \to \text{Transvers} \\ \eta^{\mu\nu} h_{\mu\nu} = 0 \to \text{Traceless} \end{array}\right\} \to h''_{\mu\nu} = h^{TT}_{\mu\nu}$$

$\langle \partial_\mu h_{\alpha\beta} \cdot \partial_\nu^* h^{\alpha\beta} \rangle$ is invariant on transformations

$$x'^\mu = x^\mu + \xi^\mu(x)$$

$$h_{\alpha\beta} \to h'_{\alpha\beta} = h_{\alpha\beta} \underbrace{- \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha}_{-\xi_{\alpha\beta}} \qquad \text{with } \partial\xi < O(h)$$

$$\langle \partial_\mu (h_{\alpha\beta} - \xi_{\alpha\beta}) \cdot \partial_\nu^* (h^{\alpha\beta} - \xi^{\alpha\beta}) \rangle = \langle \partial_\mu h_{\alpha\beta} \partial_\nu^* h^{\alpha\beta} \rangle +$$
$$- \langle \partial_\mu h_{\alpha\beta} \partial_\nu \xi^{\alpha\beta} \rangle - \langle \partial_\mu \xi_{\alpha\beta} \partial_\nu h^{\alpha\beta} \rangle$$
$$+ O(\partial\xi^2)$$

$$\langle \partial_\mu h_{\alpha\beta} \partial_\nu (\partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha) \rangle = 2 \langle \partial_\mu h_{\alpha\beta} \partial_\nu \partial^\alpha \xi^\beta \rangle \; (*)$$

$\langle \ldots \rangle$ is an integration ~~over time~~ over volume

$f(x_0, x, y, z)$ $g(x_0, x, y, z)$ $x_0 = c \cdot t$ $\Box f, g = 0 \to z = -x_0$ if propagates along z

$$\int dz\, g \cdot \partial^0 f = - \int dz\, g \cdot \partial^z f = - g \cdot f \Big|_{-Z}^{Z} + \int dz\, \partial^z g \cdot f =$$
$$= - g \cdot f \Big|_{-Z}^{Z} - \int dz\, \partial^0 g \cdot f \quad \text{therefore}$$

$$(*) = \underbrace{2 \left[\partial_\mu h_{\alpha\beta} \partial_\nu \xi^\beta \right]_{\text{surface}}}_{\text{goes to } 0} - 2 \langle \underbrace{\partial_\mu \partial^\alpha h_{\alpha\beta}}_{= 0 \text{ Lorentz gauge}} \partial_\nu \xi^\beta \rangle \to 0$$

$$\frac{c^4}{32\pi G}\langle \partial_0 h_{\alpha\beta}\, \partial_0 h^{\alpha\beta}\rangle = \frac{c^2}{32\pi G}\langle \dot{h}_{\alpha\beta}\cdot \dot{h}^{\alpha\beta}\rangle$$

$$\dot{h}_{\alpha\beta}\cdot \dot{h}^{\alpha\beta} = \mathrm{Tr}\left[\begin{pmatrix} \dot{h}_+ & \dot{h}_\times \\ \dot{h}_\times & -\dot{h}_+ \end{pmatrix}\begin{pmatrix} \dot{h}_+ & \dot{h}_\times \\ \dot{h}_\times & -\dot{h}_+ \end{pmatrix}\right] = \mathrm{Tr}\left[\begin{pmatrix} \dot{h}_+^2 + \dot{h}_\times^2 & \dots \\ \dots & \dot{h}_\times^2 + \dot{h}_+^2 \end{pmatrix}\right] =$$

$$= 2\left(\dot{h}_+^2 + \dot{h}_\times^2\right) \quad \text{hence}$$

$$t_{00} = \frac{c^2}{16\pi G}\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle$$