INSPIRAL OF COMPACT BINARIES

We start from the result of Ch.3 of M.Maggiore.

If $\begin{cases} x_0(t) = R\cos(\omega t + \frac{\pi}{2}) \\ y_0(t) = R\sin(\omega t + \frac{\pi}{2}) \\ z_0(t) = 0 \end{cases}$ then $\begin{cases} h_+ = \frac{1}{r}\frac{4G\mu\omega^2R^2}{c^4}\frac{1+\cos^2\theta}{2}\cos(2\omega t_{ret} + 2\varphi) \\ h_\times = \frac{1}{r}\frac{4G\mu\omega^2R^2}{c^4}\cos\theta\sin(2\omega t_{ret} + 2\varphi) \end{cases}$

We can choose a suitable origin of time in order to have $\varphi = 0$.

So:

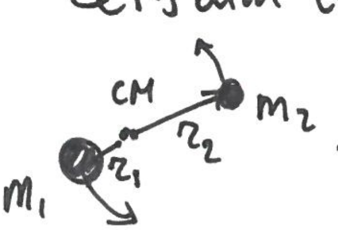
$$\begin{cases} h_+ = A \cdot \frac{1+\cos^2\theta}{2} \cdot \cos(2\omega t_{ret}) & (1) \\ h_\times = A \cdot \cos\theta \cdot \sin(2\omega t_{ret}) & (2) \end{cases} \quad \text{where } A = \frac{4G\mu\omega^2R^2}{rc^4} \quad (3)$$

the amplitude of h_+ and $h_\times$ can give the orbital plane angle θ:

$$\frac{|h_+|}{|h_\times|} = \frac{1+\cos^2\theta}{2\cos\theta} \quad (4)$$

A instead contains the product $\mu \cdot R^2$ which is a problem.

Let's aim to remove R from A.



$\rightarrow G\frac{m_1 \cdot m_2}{R^2} = m_1 \cdot r_1 \cdot \omega^2 = m_1 \cdot \frac{R \cdot m_2}{M}\omega^2$ hence

$R = r_1 + r_2$

$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 \cdot m_2}{M}$

$$\boxed{GM = R^3\omega^2} \quad \text{Kepler's Law} \quad (5)$$

The parameter A is then:

$$A = \frac{4G\mu\omega^2 R^2}{rc^4} = \frac{4G\mu\omega^2}{rc^4}(GM)^{\frac{2}{3}} \cdot \omega^{-\frac{4}{3}} = \frac{4}{rc^4} G^{\frac{5}{3}} \mu M^{\frac{2}{3}} \omega^{\frac{2}{3}} \quad (6)$$

We define the <u>CHIRP MASS</u> $\mathcal{M}_c$:

$$(7) \quad \boxed{\mathcal{M}_c^{\frac{5}{3}} = \mu \cdot M^{\frac{2}{3}}} \quad \rightarrow \quad A = \frac{1}{r}\frac{4}{c^4}(G\mathcal{M}_c)^{\frac{5}{3}} \omega^{\frac{2}{3}} \quad (8)$$

Since ω can be measured, if h_+ and $h_\times$ are accessible we can have θ and then the amplitude of h_+ or $h_\times$ depends only on $\mathcal{M}_c$ and the distance r

The degeneracy between $\mathcal{M}_c$ and r can be removed analysing the inspiral. The energy balance requires:

$$P_{GW} = -\dot{E}_{orbit} \quad (9)$$

$$E_{orbit} = K + V = \frac{1}{2}\mu R^2 \omega^2 - G\frac{m_1 m_2}{R} \overset{(5)}{=} \frac{1}{2}\mu R^2 \frac{GM}{R^3} - G\frac{m_1 m_2}{R}$$

$$= \frac{1}{2}V = -\frac{1}{2}G\frac{m_1 \cdot m_2}{R} \quad (10)$$

Therefore, $\dot{E}_{orbit} < 0 \implies \dot{R} < 0$

We still want to operate in a condition where Kepler's Law is still valid, even if $\dot{R}<0$.

Differentiating Kepler's Law (5):

$$0 = 3R^2 \delta R \cdot \omega^2 + 2R^3 \omega \delta\omega \rightarrow -3\delta R\, \omega = 2R\delta\omega$$

$$\delta R = -\frac{2}{3} R \frac{\delta\omega}{\omega} \rightarrow \dot{R} = -\frac{2}{3} R \frac{\dot{\omega}}{\omega}$$

We would like to have $\dot{R} \ll \omega R$, then:

$$\frac{2}{3} R \frac{\dot{\omega}}{\omega} \ll \omega R \Rightarrow \boxed{\frac{\dot{\omega}}{\omega^2} \ll 1} \quad (11)$$

From (9) we would like to have a diff. equation on ω, or on $\omega_{GW} = 2\omega$. We need to remove R from (9)

$$P_{GW} = \frac{32}{5} G \frac{\mu^2 R^4 \omega^6}{c^5} \overset{(5)}{=} \frac{32}{5} \frac{G}{c^5} \mu^2 (GM)^{\frac{4}{3}} \omega^{-\frac{8}{3}} \cdot \omega^6 =$$

$$= \frac{32}{5} \frac{\mu^2}{c^5} G^{\frac{7}{3}} M^{\frac{4}{3}} \omega^{\frac{10}{3}} \quad (12)$$

$$E_{orbit} = -\frac{G}{2} \frac{m_1 m_2}{R} = -\frac{G}{2} \frac{M\mu}{R} \overset{(5)}{=} -\frac{G}{2} M\mu\, \omega^{\frac{2}{3}} (GM)^{-\frac{1}{3}} =$$

$$= -\frac{1}{2} G^{\frac{2}{3}} \mu M^{\frac{2}{3}} \omega^{\frac{2}{3}} \quad (13)$$ differentiating

$$\dot{E}_{orbit} = -\frac{1}{3} G^{\frac{2}{3}} \mu M^{\frac{2}{3}} \omega^{-\frac{1}{3}} \dot{\omega} \quad (14)$$

Now we can write again eq. (9):

$$\frac{32}{5}\frac{\mu^2}{c^5}G^{\frac{7}{3}}M^{\frac{4}{3}}\omega^{\frac{10}{3}} = +\frac{1}{3}\mu G^{\frac{2}{3}}M^{\frac{2}{3}}\omega^{-\frac{1}{3}}\dot{\omega}$$

$$\frac{\dot{\omega}}{\omega^{\frac{11}{3}}} = +\frac{96}{5}\frac{1}{c^5}G^{\frac{5}{3}}\mu M^{\frac{2}{3}} = +\frac{96}{5}\frac{1}{c^5}\left(G\mathcal{M}_c\right)^{\frac{5}{3}}$$

Let's replace ω with ω_{GW}

$$\dot{\omega}\cdot\omega^{-\frac{11}{3}} = \frac{\dot{\omega}_{GW}}{2}\cdot\omega_{GW}^{-\frac{11}{3}}\cdot 2^{\frac{11}{3}} = \frac{\dot{\omega}_{GW}}{\omega_{GW}^{\frac{11}{3}}}\cdot\frac{2^{3}}{2^{\frac{1}{3}}}$$ hence

$$\boxed{\frac{\dot{\omega}_{GW}}{\omega_{GW}^{\frac{11}{3}}} = +\frac{12}{5}\sqrt[3]{2}\,\frac{1}{c^5}\left(G\mathcal{M}_c\right)^{\frac{5}{3}}} \quad (15)$$