THE GEODESIC AND THE GEODESIC DEVIATION EQUATIONS

Let's take a timelike curve $\mathcal{C}(\lambda)$, i.e. a curve where

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu < 0 \quad (1)$$

ds^2 is a main invariant of the GR. An observer following the line $\mathcal{C}(\lambda)$ will have $dx^i = 0$ so that $dx^0 = c\,d\tau$ where τ is the proper time. The previous equation then reads:

$$-c^2 d\tau^2 = g_{\mu\nu}\, dx^\mu dx^\nu = g_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} d\lambda^2 \quad (2)$$

It seems convenient to choose τ as the parameter of the curve $\mathcal{C}(\lambda)$ so that the previous equation reads:

$$c^2 = -g_{\mu\nu}\, u^\mu u^\nu \quad \text{where } u^\mu = \frac{dx^\mu}{d\tau} \quad (3)$$

u^μ are the components of the 4-vector speed $\vec{U}$.
Since τ has been defined from (1), τ is an invariant under any coordinate trasformation.
The 4-vector speed $\vec{U}$ is defined as:

$$\vec{U} = \frac{dx^\mu}{d\tau}\hat{e}_\mu \quad (4)$$

where $\hat{e}_\mu$ is the local basis of unitary 4-vectors.

Now, let's recal the total derivation of a vector.

Be $\vec{V} = V^{\gamma}\hat{e}_{\gamma}$ a 4-vector.

Let's move the origin of our spacetime by dx^{α}.

The total derivative of the 4-vector is:

$$\frac{\nabla \vec{V}}{\nabla x^{\alpha}} = \frac{dV^{\gamma}}{dx^{\alpha}} \cdot \hat{e}_{\gamma} + V^{\gamma} \cdot \frac{d\hat{e}_{\gamma}}{dx^{\alpha}} \qquad (5)$$

Moving ~~from~~ the origin the basis change from $\hat{e}_{\mu}$ to $\hat{e}'_{\mu}$. The Christoffel symbols are there to give us the law of variation of the basis.

(5) becomes:

$$\frac{\nabla \vec{V}}{\nabla x^{\alpha}} = \frac{dV^{\gamma}}{dx^{\alpha}} \hat{e}_{\gamma} + V^{\gamma} \Gamma^{\nu}_{\gamma\alpha} \hat{e}_{\nu} = \text{changing } \gamma \to \nu$$

$$= \frac{dV^{\nu}}{dx^{\alpha}} \hat{e}_{\nu} + V^{\gamma} \Gamma^{\nu}_{\gamma\alpha} \hat{e}_{\nu} = \left(\frac{dV^{\nu}}{dx^{\alpha}} + V^{\gamma} \Gamma^{\nu}_{\gamma\alpha} \right) \hat{e}_{\nu} \quad (6)$$

Therefore we can define the total derivative of the components of a vectors as:

$$\frac{\nabla V^{\nu}}{\nabla x^{\alpha}} = \frac{dV^{\nu}}{dx^{\alpha}} + V^{\gamma} \Gamma^{\nu}_{\gamma\alpha} \qquad (7)$$

If we consider a curve $\mathcal{C}(\lambda)$ then the total derivative of a vector along the curve is:

$$\frac{\nabla \vec{V}}{\nabla \lambda} = \left(\frac{\nabla V^{\nu}}{\nabla \lambda}\right) \hat{e}_{\nu} \quad \text{where} \quad \frac{\nabla V^{\nu}}{\nabla \lambda} = \frac{dV^{\nu}}{dx^{\alpha}}\frac{dx^{\alpha}}{d\lambda} + V^{\gamma}\Gamma^{\nu}_{\gamma\alpha}\frac{dx^{\alpha}}{d\lambda} \quad (8)$$

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Now, let's take a curve $\mathcal{C}(\lambda)$ that is timelike, i.e. $ds^2<0$. Let's fix 2 points, A and B, on that curve. The time-law that defines the kinematics of a massive point along that trajectory $\mathcal{C}(\lambda)$ is the one that extremises the action S defined as:

$$S = -m\int_{\tau_A}^{\tau_B} d\tau \quad (9)$$

The proper time is well defined since $ds^2<0$ on $\mathcal{C}(\lambda)$. The formal procedre is to consider the Euler-Lagrange equation and solve the problem. Insted I justify the result, following a simple argoment.

In the action (9) there are not any potentials therefore it seems intuitive that the kinematics is that of a free particle: $\frac{\nabla \vec{U}}{\nabla \tau}=0$ (not $\frac{d\vec{U}}{d\tau}=0$)

where $\vec{U}$ is the velocity 4-vector: $\vec{U} = \frac{dx^\mu}{d\tau} \cdot \hat{e}_\mu$ (10)

$= U^\mu \hat{e}_\mu$

Hence from (8)

$$\frac{\nabla u^\mu}{\nabla \tau} = \frac{du^\mu}{d\tau} + u^\gamma \Gamma^\mu_{\gamma\nu} \frac{dx^\nu}{d\tau} = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\gamma\nu} u^\gamma u^\nu \quad (11)$$

So, the GEODESIC EQUATION is:

$$\boxed{\frac{du^\mu}{d\tau} + \Gamma^\mu_{\gamma\nu} u^\gamma u^\nu = 0} \quad (12)$$

Now, let's imagine to have 2 curves, both respect the geodesic equation. The first is described by $x^\mu(\tau)$ the other by $x^\mu(\tau) + \xi^\mu(\tau)$, i.e. the second is a curve described by the first observer.

If we apply the (12) on $\frac{d}{d\tau}(x^\mu + \xi^\mu)$ we have:

$$\frac{d^2}{d\tau^2}(x^\mu + \xi^\mu) + \Gamma^{\mu'}_{\gamma\nu}\left(\frac{dx^\gamma}{d\tau} + \frac{d\xi^\gamma}{d\tau}\right)\left(\frac{dx^\nu}{d\tau} + \frac{d\xi^\nu}{d\tau}\right) =$$

$$= \frac{d^2\xi^\mu}{d\tau^2} + \frac{d^2x^\mu}{d\tau^2} + \left[\Gamma^\mu_{\gamma\nu} + (\partial_\alpha \Gamma^\mu_{\gamma\nu}) \cdot \xi^\alpha\right](\quad)^\gamma(\quad)^\nu$$

we suppose that ξ^μ is "small"

Considering that $x^\mu(\tau)$ respects a geodesic equation:

$$\frac{d^2\xi^\mu}{d\tau^2} + \left[\Gamma^\mu_{\gamma\nu} + \xi^\alpha\partial_\alpha\Gamma^\mu_{\gamma\nu}\right]\frac{d\xi^\gamma}{d\tau}\frac{d\xi^\nu}{d\tau} + \xi^\alpha\partial_\alpha\Gamma^\mu_{\gamma\nu}\frac{dx^\gamma}{d\tau}\frac{dx^\nu}{d\tau} +$$

$$+ \left[\Gamma^\mu_{\gamma\nu} + \xi^\alpha\partial_\alpha\Gamma^\mu_{\gamma\nu}\right]\left(\frac{dx^\gamma}{d\tau}\frac{d\xi^\nu}{d\tau} + \frac{d\xi^\gamma}{d\tau}\frac{dx^\nu}{d\tau}\right) = 0$$

Considering $\frac{d\xi^\mu}{d\tau}$ small as ξ^μ, taking only the terms proportional to either ξ^μ or $\frac{d\xi^\mu}{d\tau}$, we have:

$$\boxed{\frac{d^2\xi^\mu}{d\tau^2} + \xi^\alpha\left(\partial_\alpha\Gamma^\mu_{\gamma\nu}\right)\frac{dx^\gamma}{d\tau}\frac{dx^\nu}{d\tau} + 2\,\Gamma^\mu_{\gamma\nu}\frac{dx^\gamma}{d\tau}\frac{d\xi^\nu}{d\tau} = 0} \quad (13)$$

This is the GEODESIC DEVIATION EQUATION