The Metric of a Free Falling Frame

We have seen that the possibility to have a flat metric in the neighbours of a point P (possibility goranteed by the equivalence principle) can be extendend ~~to~~ all along a geodesic. One can use the Fermi coordinates and with that along a geodesic:

$$ds^2 = -c^2 dt^2 + \delta_{ij} dx^i dx^j \qquad (1)$$

If we move x^i from the origin, how the metric would change? Flat metric implies that the Γs are zero ~~a~~ on the origin but since the Γs are derivatives of $g_{\mu\nu}$ on the origin we have $\partial_i g_{\mu\nu} = \partial_i \eta_{\mu\nu} = 0$

Therefore

$$\delta \eta_{\mu\nu} = \partial_i \eta_{\mu\nu} \cdot \delta x^i = 0 \qquad (2)$$

which means we cannot have corrections of (1) linear in the distance x^i from the origin. ~~of (1)~~.

It can be demostrated that the correction of (1) at the second order of r (i.e. $O(r^2)$) are:

$$ds^2 = -c^2dt^2\left[1 + R_{0i0j}\,x^ix^j\right] +$$
$$+ dx^idx^j\left[\delta_{ij} - \frac{1}{3}R_{ikjl}\,x^kx^l\right] +$$
$$-2c\,dt\,dx^i\left[\frac{2}{3}R_{0jik}\cdot x^jx^k\right] \qquad (3)$$

Since $R_{\mu\nu\rho\sigma}$ is the $\partial^2 g_{\mu\nu}$ then $[R_{\mu\nu\rho\sigma}] = m^{-2}$ and also $R_{\mu\nu\rho\sigma} \sim \frac{1}{L_B^2}$ where L_B is a typical lengthscale. All the correction terms in (3) are of the order $\left(\frac{r_c}{L_B}\right)^2$.

SYMMETRIES

$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$

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Forcing a bit the notation we can write:

$$\delta\eta_{00} = -R_{0i0j}\,x^ix^j \qquad (4)$$
$$\delta\eta_{0i} = -\frac{2}{3}R_{0jik}\,x^jx^k \quad (5) \qquad \delta\eta_{0i} = \delta\eta_{i0}$$
$$\delta\eta_{ij} = -\frac{1}{3}R_{ikjl}\,x^kx^l \quad (6)$$

So, from the flat metric (1) and the first correction (3) we can write:

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta\eta_{\mu\nu} \qquad (7)$$

The linearized Christoffel symbols are:

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}\left(\partial_\nu \delta\eta_{\rho\sigma} + \partial_\rho \delta\eta_{\nu\sigma} - \partial_\sigma \delta\eta_{\nu\rho}\right) =$$

$$= \frac{1}{2}\left(\partial_\nu \delta\eta_\rho{}^{\mu} + \partial_\rho \delta\eta_\nu{}^{\mu} - \partial^\mu \delta\eta_{\nu\rho}\right)$$

The derivatives act on the Rs and on the xs in the (4) to (6) expressions but the only hope to produce terms that are in the lowest order possible in $\frac{r}{L_B}$ is letting the derivatives acting on the xs.

In that way one can see that the leading term of Γs is of the order $\frac{r}{L_B^2}$

Still, in the geodesic deviation equation (13), the last term is infinitesimal of the order $\frac{r}{L_B^2}$, but the second term, since it is proportional to $\partial\Gamma$, it has the chance to be finite and not infinitesimal

Considering that $\frac{dx^i}{d\tau} = 0$, let's see only the term:

$$\xi^j\left(\partial_j \Gamma^i_{00}\right)\left(\frac{dx^0}{d\tau}\right)^2 = \frac{1}{2}\left(\frac{dx^0}{d\tau}\right)^2 \xi^j \partial_j\left(\overset{\dot{x}^i=0}{\overset{\uparrow}{\partial_0 \delta\eta_0{}^i}} + \overset{\dot{x}^i=0}{\overset{\uparrow}{\partial_0 \delta\eta_0{}^i}} - \partial^i \delta\eta_{00}\right)$$

(*) $\longrightarrow$ next page

$$= \frac{1}{2}\left(\frac{dx^0}{d\tau}\right)^2 \xi^j \left(\overset{(*)}{2R_0{}^i{}_{0j}}\right) \simeq c^2 R_0{}^i{}_{0j}\,\xi^j.$$

In the free falling frame, the leading terms in the geodesic deviation equation are:

$$\boxed{\frac{d^2\xi^i}{d\tau^2} = - c^2 R_0{}^i{}_{0j}\,\xi^j} \qquad (8)$$

$$\boxed{\begin{aligned} (*)\quad & -\partial_j \partial^i \delta\eta_{00} \overset{(4)}{=} + R_{0l0k}\,\partial_j \partial^i (x^l \cdot x^k) = \\ & = R_{0l0k}\,\partial_j (\delta^{il} x^k + x^l \delta^{ik}) = \\ & = R_{0l0k}\,(\delta^{il}\delta_j{}^k + \delta_j{}^l \delta^{ik}) = R_0{}^i{}_{0j} + R_{0j0}{}^i = \\ & = R_0{}^i{}_{0j} + R_0{}^i{}_{0j} = 2 R_0{}^i{}_{0j} \end{aligned}}$$

thanks to the symmetries of $R^\mu{}_{\nu\rho\sigma}$, eq (8) can be also written as

$$\frac{d^2\xi^i}{d\tau^2} = - c^2 R^i{}_{0j0}\,\xi^j$$

It can be shown that the Riemann tensor is gauge invariant. So, let's calculate $R^i{}_{0j0}$ in the TT gauge:

$$R^{i}{}_{0j0} = \partial_j \Gamma^{i}_{00} - \partial_0 \Gamma^{i}_{0j}$$ knowing that:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu h^{\rho}{}_{\nu} + \partial_\nu h^{\rho}{}_{\mu} - \partial^{\rho} h_{\mu\nu}\right)$$ we have

$$\Gamma^{i}_{00} = \frac{1}{2}\left(\partial_0 h^{i}{}_{0} + \partial_0 h^{i}{}_{0} - \partial^{i} h_{00}\right) = 0$$ in the TT gauge.

$$\Gamma^{i}_{0j} = \frac{1}{2}\left(\partial_0 h^{i}{}_{j} + \partial_j h^{i}{}_{0} - \partial^{i} h_{0j}\right) = \frac{1}{2}\partial_0 h^{i}{}_{j}$$

The Riemann tensor is then:

$$R^{i}{}_{0j0} = -\frac{1}{2}\partial_0^2 h^{i}{}_{j} = -\frac{1}{2c^2}\ddot{h}^{TT\,i}{}_{j} \quad (9)$$

the geodesic deviation equation (8) becomes:

$$\boxed{\frac{d^2\xi^i}{d\tau^2} = \frac{1}{2}\ddot{h}^{TT\,i}{}_{j}\,\xi^j} \quad (10)$$