The effective Einstein's equation and the GW E-M tensor

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) \quad (1)$$

Once we consider $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ we have

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \dots \quad (2)$$

Now we calculate the effective tensor:

$$\langle R_{\mu\nu} \rangle = \bar{R}_{\mu\nu} + \langle R^{(2)L}_{\mu\nu} \rangle \quad (3)$$ hence:

$$\bar{R}_{\mu\nu} = -\langle R^{(2)L}_{\mu\nu} \rangle + \frac{8\pi G}{c^4}\left\langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right\rangle \quad (4)$$

We define $t_{\mu\nu} = -\frac{c^4}{8\pi G}\left\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} R^{(2)} \right\rangle^{(5)}$ where

$$R^{(2)} = \bar{g}^{\mu\nu} R^{(2)}_{\mu\nu} \rightarrow t = \bar{g}^{\mu\nu} t_{\mu\nu} = -\frac{c^4}{8\pi G}\left\langle R^{(2)} - \frac{1}{2} 4 R^{(2)} \right\rangle =$$

$$= \frac{c^4}{8\pi G}\langle R^{(2)} \rangle \quad (6)$$

Replacing (6) into (5) we have

$$t_{\mu\nu} = -\frac{c^4}{8\pi G}\langle R^{(2)}_{\mu\nu} \rangle + \bar{g}_{\mu\nu} t \qquad \text{i.e.:}$$

$$\langle R^{(2)}_{\mu\nu} \rangle = -\frac{8\pi G}{c^4}\left(t_{\mu\nu} - \bar{g}_{\mu\nu} t\right) \quad (7)$$

Replacing (7) in (4):

$$\bar{R}_{\mu\nu} = \frac{8\pi G}{c^4}\left[\left(t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} t\right) + \left(\bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}\right)\right] \quad (8)$$

Once again, multipling by $\bar{g}_{\mu\nu}$ we have:

$$\bar{R} = \frac{8\pi G}{c^4}\left[-t - \bar{T}\right]$$ hence

$$\bar{R}_{\mu\nu} = \frac{8\pi G}{c^4}\left(\bar{T}_{\mu\nu} + t_{\mu\nu}\right) + \frac{1}{2}\bar{g}_{\mu\nu}\frac{8\pi G}{c^4}\left(t + \bar{T}\right)$$ i.e.

$$\boxed{\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\cdot\bar{R} = \frac{8\pi G}{c^4}\left(\bar{T}_{\mu\nu} + t_{\mu\nu}\right)} \quad (9)$$

This is the low-frequency part of the Einstein's equation, otherwise called the coarse-grained equation.

The E-M tensor for GW is then

$$t_{\mu\nu} = -\frac{c^4}{8\pi G}\left[\langle R_{\mu\nu}^{(2)}\rangle - \frac{1}{2}\bar{g}_{\mu\nu}\langle R^{(2)}\rangle\right] \quad (5)$$

After long calculations and considering the TT gauge we have

$$\langle R_{\mu\nu}^{(2)}\rangle = -\frac{1}{4}\langle \partial_\mu h_{\alpha\beta}\,\partial_\nu h^{\alpha\beta}\rangle \quad (10)$$

First consequence:

$$R^{(2)} = \langle g^{\mu\nu} R^{(2)}_{\mu\nu} \rangle = -\frac{1}{4} \langle \partial^\nu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \rangle =$$

$$= -\frac{1}{4} \Big\{ \underbrace{\int h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \cdot n^\nu \, dS}_{\text{srcfree} \,\to\, =0} \ast \langle \underbrace{\partial^\nu \partial_\nu h_{\alpha\beta}}_{\Box h_{\alpha\beta} = 0} \, h^{\alpha\beta} \rangle = \qquad (11)$$

$$= 0$$

So,

$$t_{\mu\nu} = \frac{c^4}{32\pi G} \langle \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \rangle \qquad (12)$$

$$t_{00} = \frac{c^2}{32\pi G} \langle \dot{h}^{TT}_{\alpha\beta} \cdot \dot{h}^{\alpha\beta} \rangle =$$

$$= \frac{c^2}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle \qquad (13)$$