Effect of a GW on an interferometer in the TT gauge

We consider ~~the Free Fall Frame (at least in the horizontal plane the test masses are free.)~~ ~~and~~ the TT gauge.

Let's imagine that a GW with polarization + propagates along the axe z. It is not important the direction because the only relevant fact is the value of h on the plane where the test masses are located ($z=0$). The Michelson interferometer has the arms along the x and y axes.

For 2 events separated by an infinitesimal distance on the detector plane we have

$$ds^2 = -c^2dt^2 + [1+h_+(t)]dx^2 + [1-h_+(t)]dy^2 \quad (1)$$

t, x, and y are the coordinated as seen by the observer using ~~[illegible]~~ the TT gauge.

A photon has $ds^2 = 0$.

For a photon that propagates along the x-axis:

$$c^2 dt^2 = [1 + h_+] dx^2 \quad (2)$$

which means $dt \cong \pm \left[1 + \frac{h_+}{2}\right] \frac{dx}{c}$ (3) considering that $h_+ \ll 1$. $+$ is for ~~waves~~ photons travelling right bound whereas $-$ is for left bound photons.

Let's calculate the time the photon takes to go to the mirror and come back.

$$t_2 - t_0 = t_2 - t_1 + t_1 + t_0 = \int_{t_0}^{t_1} dt + \int_{t_1}^{t_2} dt =$$

$$= \int_0^{Lx} \left[1 + \frac{h_+}{2}\right] \frac{dx}{c} - \int_{Lx}^{0} \left[1 + \frac{h_+}{2}\right] \frac{dx}{c} = \frac{2Lx}{c} + \int_0^{Lx} dx - \int_{Lx}^{0} dx$$

Now a trick: from (3), to 0-order in h, $\frac{dx}{c} = \pm dt$

hence:

The position L_x of the mirror does not change because we are in the TT gauge

$$\cancel{t_2 - t_0 = 2 \int_{t_0}^{Lx/c} \left[1 + \frac{h_+}{2}\right] dt}$$

$$t_2 - t_0 = \frac{2Lx}{c} + \frac{1}{2} \int_{t_0}^{Lx/c} h_+(t) dt + \frac{1}{2} \int_{\frac{Lx}{c}}^{t_2} h_+(t) dt =$$

$$= \frac{2L_x}{c} + \frac{1}{2}\int_{t_0}^{t_2} h_+(t)\,dt = \frac{2L_x}{c} + \frac{1}{2} h_0 \int_{t_0}^{t_2} \sin(\Omega t)\,dt$$

$$t_2 - t_0 = \frac{2L_x}{c} + \frac{h_0}{2\Omega}\left[\cos(\Omega t_0) - \cos(\Omega t_2)\right] \quad (4)$$

at 0-order in h we have $t_2 = t_0 + \frac{2L_x}{c}$ hence :

$$t_2 - t_0 = \frac{2L_x}{c} + \frac{h_0}{2\Omega}\left[\cos(\Omega t_0) - \cos\left\{\Omega\left[t_0 + \frac{2L_x}{c}\right]\right\}\right]$$

$$\cos\alpha - \cos(\alpha + 2\beta) = \cos\alpha - \cos\alpha\cos 2\beta + \sin\alpha\sin 2\beta =$$

$$= \cos\alpha(1 - \cos 2\beta) + \sin\alpha \sin 2\beta = 2\cos\alpha \sin^2\beta + 2\sin\alpha\sin\beta\cos\beta =$$

$$= 2\sin\beta(\cos\alpha\sin\beta + \sin\alpha\cos\beta) = 2\sin\beta \sin(\alpha+\beta)$$

Hence :

$$t_2 - t_0 = \frac{2L_x}{c} + \frac{h_0}{\Omega}\sin\left[\Omega\left(t_0 + \frac{L_x}{c}\right)\right]\cdot \sin\left(\Omega\frac{L_x}{c}\right) =$$

$$= \frac{2L_x}{c}\left\{1 + \frac{h_0}{2}\sin\left[\Omega\left(t_0 + \frac{L_x}{c}\right)\right]\cdot \frac{\sin\left(\Omega\frac{L_x}{c}\right)}{\Omega\frac{L_x}{c}}\right\} \quad (5)$$

the previous equation needs to be explained a bit.
t_0 and t_2 are 2 running times, one linked to
the beam splitter when the beam departs and the
other, t_2, linked to the BS when the beam arrives.
At the departure time the electric field is $\vec{E}_0 e^{-i\omega t_0}$
At the arrival time the electric field is $\frac{\vec{E}_0}{2} e^{-i\omega t_2}$
The MM book decided to work with the arrival time
then the electric field then:

$$\vec{E}^{(x)}(t) = \frac{\vec{E}_0}{2} e^{-i\omega t_0(t)} = \frac{\vec{E}_0}{2} e^{-i\omega\left\{t - \frac{2L_x}{c} + \frac{h_0}{\Omega} \sin\left[\Omega\left(t - \frac{L_x}{c}\right)\right] \sin\frac{\Omega L_x}{c}\right\}} \quad (6)$$

where we made the replacement of t_2 with t
and, to zero order in h, we have $t_2 = t = t_0 + \frac{2L_x}{c}$
The previous can be written as:

$$\vec{E}^{(x)}(t) = \frac{\vec{E}_0}{2} e^{-i\left\{\omega\left(t - \frac{2L_x}{c}\right) - \delta\varphi^{(x)}\right\}} \quad (7)$$

where $\delta\varphi^{(x)} = \frac{\omega}{\Omega} h\left(t - \frac{L_x}{c}\right) \cdot \sin\left(\Omega \frac{L_x}{c}\right)$ (8)

With similar arguments we have

$$\vec{E}^{(y)}(t) = -\frac{\vec{E}_0}{2}\, e^{-i\left\{\omega\left(t-\frac{2L_y}{c}\right)+\delta\varphi^{(y)}\right\}} \quad (9)$$

where $$\delta\varphi^{(y)} = \frac{\omega}{\Omega} h\left(t-\frac{L_y}{c}\right)\cdot \sin\left(\Omega\frac{L_y}{c}\right) \quad (10)$$

So, the electric field leaving the BS towards the photodiode is:

$$\vec{E}_{OUT} = \frac{\vec{E}_0}{2}\left\{ e^{-i\left[\omega\left(t-\frac{2L_x}{c}\right)-\delta\varphi^{(x)}\right]} - e^{-i\left[\omega\left(t-\frac{2L_y}{c}\right)+\delta\varphi^{(y)}\right]}\right\} \quad (11)$$

The intensity is proportional to $\vec{E}^*\cdot\vec{E}$.

$$I \propto \vec{E}_{out}\cdot\vec{E}_{OUT}^{*} = \frac{E_0^2}{4}\left\{ 1+1 - e^{-i\left[\omega\frac{2(L_y-L_x)}{c}-\delta\varphi^{(x)}-\delta\varphi^{(y)}\right]} - e^{+i\left[\omega\frac{2(L_y-L_x)}{c}-\delta\varphi^{(x)}-\delta\varphi^{(y)}\right]}\right\} =$$

$$= \frac{E_0^2}{2}\left\{1-\cos\left[\omega\frac{2(L_y-L_x)}{c}-\delta\varphi^{(x)}-\delta\varphi^{(y)}\right]\right\} \quad (12)$$

The length difference is small compared to the lengths, so:

$$L_y - L_x = \Delta L \ll L_x, L_y \rightarrow \delta\varphi^{(x)}+\delta\varphi^{(y)} = 2\delta\varphi = 2\frac{\omega}{\Omega} h_0 \sin\left(t-\frac{L}{c}\right)\sin\left(\frac{\Omega L}{c}\right)$$

$$I \propto \frac{E_0^2}{2}\left\{1-\cos\left[\varphi_0 - 2\delta\varphi\right]\right\}$$

$$\delta\varphi = \frac{\omega}{\Omega} h_0 \cdot \sin\left[\Omega\left(t-\frac{L}{c}\right)\right] \cdot \sin\left(\frac{\Omega L}{c}\right)$$

$\delta\varphi \sim \frac{10^{14}}{10^{2}} \cdot 10^{-21} = 10^{-9}$ rad. $\ll 1$ then

$$I \propto \frac{E_0^2}{2}\left\{1-\cos\varphi_0 - \sin\varphi_0 \cdot 2\delta\varphi\right\} \quad (13)$$

It seems better to have $\varphi_0 = \frac{\pi}{2}$ but the DC is huge!

Instead, if φ_0 is close to 0 we can write:

$$I \propto \frac{E_0^2}{2}\left\{\cancel{1} - \cancel{1} + \frac{\varphi_0^2}{2} - \varphi_0 \cdot 2\delta\varphi\right\}$$

Hence, the signal-to-DC ratio is $\frac{2\delta\varphi}{\varphi_0}$

If one consider the signal-to-noise ratio, the main source of noise is the shot noise, which is proportional to the square root of the λ density $\rightarrow I_{shot} \propto \varphi_0$

hence the SNR become constant for $\varphi_0 \rightarrow 0$