Response of an interferometer to GW in the FFF ①

We have the Michelson Interferometer with the ~~axis~~ arms along the axis x and y.
The beamsplitter, the two mirrors at the end of the arm are in the free-falling condition. This condition is met by the 3 masses for displacements on the x-y plane and at frequencies above few Hz.
the observer, located on the beamsplitter, can choose a coordinate system so that the metric is perfectly flat. and it will continue to be flat all along its geodesic.
The directions of the axis are kept with the gyroscopes.
The infinitesimal interval is

$$ds^2 = -c^2dt^2 + \delta_{ij}\, dx^i dx^j \qquad (1)$$

but when we move far from the observer the space-time start to show the curvature. We have demonstated that the variation $\delta\eta_{\mu\nu}$ cannot be linear on the coordinates otherwise locally the metric cannot be flat.
So, at the first order on $(r/L_B)^2$, we have

$$ds^2 = -c^2dt^2\left[1 + R_{0i0j}\cdot x^i\cdot x^j\right] +$$
$$- 2c\,dt\,dx^i\left[\frac{2}{3}R_{0jik}\cdot x^j\cdot x^k\right] +$$
$$+ dx^i\cdot dx^j\left[\delta_{ij} - \frac{1}{3}R_{ijkl}\cdot x^k\cdot x^l\right] \qquad (2)$$

L_B is the typical variation length of the metric.

In case of the presence of GW the typical length scale is $\lambda\!\!\!^{-}_{GW} = \frac{\lambda}{2\pi} = \frac{T \cdot c}{2\pi} = \frac{c}{\omega_{GW}}$

For 100 Hz we have $\lambda\!\!\!^{-}_{GW} = \frac{c}{2\pi f} = \frac{3 \cdot 10^8}{2\pi \cdot 10^2} \simeq 500$ km

Future GW detector will have a maximum dimension of 40 km so, our calculations are correct to about $1\% \simeq \left(\frac{40}{500}\right)^2$

So, let's imagine to have a photon that propagates along the axes x. The other 2 coordinates are zero as well as the differential dy and dz. Being a photon, $ds^2 = 0$ and then eq. (2) becomes:

$$(2): \quad 0 = -c^2 dt^2 \left[1 + R_{0x0x} \cdot x^2\right] + \\ -2c\,dt\,dx \left[\frac{2}{3} R_{0xxx} \cdot x^2\right] + \\ + dx^2 \left[1 - \frac{1}{3} R_{xxxx} \cdot x^2\right] \quad (3)$$

Since the Riemann tensor is antisymmetric on the exchange of the first an second pair of indeces, we have:

$$(2): \quad 0 = -c^2 dt^2 \left[1 + R_{0x0x} \cdot x^2\right] + dx^2 \quad (4)$$

At the first order in h we have $R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho}$

then $R^{0}{}_{x0x} = \partial_0 \Gamma^{0}_{xx} - \partial_x \Gamma^{0}_{x0}$ and considering that:

$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} \eta^{\rho\sigma} (\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}) + O(h^2)$ we have:

$$\Gamma^0_{xx} = \frac{1}{2}\eta^{0\sigma}(\partial_x h_{\sigma x} + \partial_x h_{\sigma x} - \partial_\sigma h_{xx}) = \frac{1}{2}(\partial_x h^0_x \cdot 2 - \partial^0 h_{xx})$$

$$\Gamma^0_{x0} = \frac{1}{2}\eta^{0\sigma}(\partial_x h_{\sigma 0} + \partial_0 h_{\sigma x} - \partial_\sigma h_{x0}) = \frac{1}{2}(\partial_x h^0_0 + \partial_0 h^0_x - \partial^0 h_{x0})$$

The textbook says that the Riemann tensor is invariant in the linearized theory so, we can consider the GW in the TT gauge. Therefore we have only 1 surviving term, the h_{xx} :

$$\left.\begin{aligned} \Gamma^0_{xx} &= -\frac{1}{2}\partial^0 h_{xx} = \frac{1}{2}\partial_0 h^+ \\ \Gamma^0_{x0} &= 0 \end{aligned}\right\} (5)$$

with those we can compute the $R^0{}_{x0x}$

$$R^0{}_{x0x} = \partial_0 \Gamma^0_{xx} = \frac{1}{2}\partial_0^2 h^+ \quad (6) \qquad R_{0x0x} = -\frac{1}{2}\partial_0^2 h^+$$

Equation (4) becomes :

$$(4): \quad 0 = -c^2 dt^2\left[1 - \frac{1}{2c^2}\ddot{h}^+ \cdot x^2\right] + dx^2 \quad (7)$$

Thus $dt^2 \cong \frac{dx^2}{c^2}\left[1 + \frac{1}{2c^2}\ddot{h}^+ x^2\right]$ hence

$$dt \simeq \pm \frac{dx}{c}\left[1 + \frac{1}{4c^2}\ddot{h}^+ x^2\right] \quad (8)$$

We can use this relation to calculate the round trip time

$$t_2 - t_0 = \int_{t_0}^{t_2} dt = \int_{t_0}^{t_1} dt + \int_{t_1}^{t_2} dt = \int_0^{L_x(h)}\left[1 + \frac{\ddot{h}^+}{4c^2}x^2\right]\frac{dx}{c} - \int_{L_x(h)}^{0}\left[1 + \frac{\ddot{h}^+}{4c^2}x^2\right]\frac{dx}{c} \quad (9)$$

In the FFF the test masses (mirrors of the interferometer) feel the passage of the GW. Let's do here again the calculation of their motion.

Be $\vec{\xi}$ the separation between the beamsplitter and one of the end mirrors. The Geodesic Deviation Equation reads:

$$\frac{d^2\xi^\mu}{d\tau^2} + \xi^\alpha \left(\partial_\alpha \Gamma^\mu_{\gamma\nu}\right)\frac{dx^\gamma}{d\tau}\frac{dx^\nu}{d\tau} + 2\,\Gamma^\mu_{\gamma\nu}\frac{dx^\gamma}{d\tau}\cdot\frac{d\xi^\nu}{d\tau} = 0 \quad (10)$$

x^α are the coordinates of the geodesic taken as reference. In our case we can consider that x^α are the coordinates of the beam splitter, where the observer is located, hence:

$$x^\alpha = (c\tau, \vec{0}) \qquad (11)$$

Replacing (11) in (10) one has:

$$\frac{d^2\xi^\mu}{d\tau^2} + \xi^\alpha\left(\partial_\alpha \Gamma^\mu_{00}\right)\cdot c\cdot c + 2\,\Gamma^\mu_{0\nu}\cdot c\cdot\frac{d\xi^\nu}{d\tau} = 0 \quad (12)$$

$$\cancel{\partial_\alpha \Gamma^\mu_{00} = \partial_\alpha\left[\frac{1}{2}\eta^{\mu\sigma}\left(\partial_0 h_{\sigma 0} + \partial_0 h_{\sigma 0} - \partial_\sigma h_{00}\right)\right]} =$$

$$\cancel{= \frac{1}{2}\partial_\alpha\left(2\partial_0 h^\mu{}_0 - \partial^\mu h_{00}\right)}$$

Both the Christoffel symbol and its derivative are evaluated at x^μ. Due to (11) they are both evaluated at the origin, but at the origin, where the observer is placed, the metric is flat, hence the Christoffel symbol is null. The last term in eq. (12) is zero.

let's evaluate $\partial_\alpha \Gamma^\mu_{00}$.

In the neighbour of the origin the metric can be written as:

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta\eta_{\mu\nu} \quad (13)$$

where $\delta\eta_{\mu\nu}$ can be found from eq. (2)

$$\cancel{\delta\eta_{\mu\nu} = -R_{0i0j}\, x^i x^j \delta_{0\mu}\delta_{0\nu} - \frac{4}{3} R_{0j\nu k}\, x^j x^k \delta_{0\mu}}$$
$$\cancel{-\frac{2}{3} x^j x^k \left[R_{0j\nu k}\, \delta_{0\mu} + R_{0j\mu k}\, \delta_{0\nu} \right]}$$

In equation (2) all the Riemann tensor components are evaluated at the origin.

Either generated from GW or by the static distribution of masses we can consider $\delta\eta_{\mu\nu}$ very small.

therefore, the Christoffel symbol to the first order in $\delta\eta$ reads:

$$\Gamma^\mu_{\rho\nu} = \frac{1}{2}\eta^{\mu\sigma}\left(\partial_\rho \delta\eta_{\sigma\nu} + \partial_\nu \delta\eta_{\sigma\rho} - \partial_\sigma \delta\eta_{\rho\nu}\right) \quad \text{then:}$$

$$\Gamma^\mu_{00} = \frac{1}{2}\eta^{\mu\sigma}\left(\partial_0 \delta\eta_{\sigma 0} + \partial_0 \delta\eta_{\sigma 0} - \partial_\sigma \delta\eta_{00}\right) =$$

$$= \frac{1}{2}\eta^{\mu\sigma}\left(2\partial_0 \delta\eta_{\sigma 0} - \partial_\sigma \delta\eta_{00}\right) \quad \text{and finally:}$$

$$\partial_\alpha \Gamma^\mu_{00} = \frac{1}{2}\left(2\partial_\alpha \partial_0 \delta\eta^\mu{}_0 - \partial_\alpha \partial^\mu \delta\eta_{00}\right) \quad (14)$$

Looking at eq. (2) $\delta\eta$ is quadratic in the spatial coordinates so $\partial_\alpha \Gamma^\mu_{00}$, being evaluated at the origin, is zero unless α and μ are spatial indeces. In this case the surviving term is only the last one:

$$\partial_\alpha \Gamma^\mu_{00} = \begin{cases} 0 & \text{if } \alpha \text{ or } \mu = 0 \\ + R_{0i0j} & \text{if } \alpha \text{ and } \mu \neq 0 \end{cases} \qquad (15)$$

$$\boxed{R_{0\ell 0k}\, \partial_j \partial_i (x^k \cdot x^\ell) = = 2 R_{0i0j}}$$

The G.D.E. (12) becomes:

$$\frac{d^2 \xi^j}{d\tau^2} + \xi^i R_{0i0}{}^j \cdot c^2 = 0 \qquad (16)$$

As said in another note, the Riemann tensor components are invariant under coordinate tranformation so, we chose the TT gauge so that $R_{0i0j} = -\frac{1}{2c^2} \ddot{h}^{TT}_{ij}$, hence:

$$\ddot{\xi}^x = \frac{1}{2} \ddot{h}^+ \xi^x \qquad (17)$$

We can solve (17) at first order in h, i.e.

$$\ddot{\xi}^x \cong \frac{1}{2} \ddot{h}^+ L^x \Rightarrow \xi^x = L_x + \frac{h_0^+ L_x}{2} \cos(\Omega t) \qquad (18)$$
$$= L_x + \delta L(t)$$

Now we can calculate the time a photon takes to make the round trip, i.e. eq(9).

The first integral is (see eq. 9)

$$I_1 = \int_0^{L_x+\delta L} \left[1 + \frac{\ddot{h}^+(t)}{4c^2} x^2(t)\right] \frac{dx}{c} = \int_0^{L_x} [\quad] \frac{dx}{c} + \int_{L_x}^{L_x+\delta L} 1 \, \frac{dx}{c} + O(h^2) =$$

$$= \frac{L_x}{c} + \int_{t_0}^{t_1} \frac{\ddot{h}^+}{4c^2} x^2(t)\, dt + \frac{\delta L(t_1)}{c} + O(h^2) = \Longrightarrow$$

$$I_1 = \frac{L_x}{c} + \frac{L_x}{c}\frac{h^+(t_1)}{2} + \int_{t_0}^{t_1} \frac{\ddot{h}^+}{4c^2} x^2(t)\, dt + O(h^2)$$

at zero order in h we know $x(t)$: $x(t) = c(t-t_0)$ (19)

The integral above becomes:

$$I_{1a} = \int_{t_0}^{t_1} \frac{\ddot{h}^+}{4c^2} x^2 dt = \left.\frac{\dot{h}^+}{4c^2} x^2\right|_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{\dot{h}^+}{4c^2} 2x\cdot\dot{x}\, dt =$$

$$= \frac{\dot{h}^+(t_1)}{4c^2} \cdot L_x^2 - \overset{(19)}{\overset{\downarrow}{0}} - \int_{t_0}^{t_1} \frac{\dot{h}^+}{2c} x\, dt = \left(\frac{L_x}{2c}\right)^2 \dot{h}^+(t_1) - \left.\frac{h^+}{2c} x\right|_{t_0}^{t_1} + \int_{t_0}^{t_1} \frac{h^+}{2} dt =$$

$$= \left(\frac{L_x}{2c}\right)^2 \cdot \dot{h}^+(t_1) - \frac{h^+(t_1)}{2c} L_x + \frac{h_0}{2\Omega}\left[\sin(\Omega t_1) - \sin(\Omega t_0)\right]$$

Putting all together, there is a term that dishappares:

$$I_1 = \frac{L_x}{c} + \left(\frac{L_x}{2c}\right)^2 \cdot \dot{h}^+(t_1) + \frac{h_0}{2\Omega}\left[\sin(\Omega t_1) - \sin(\Omega t_0)\right] \quad (20)$$

Now we calculate the backward path duration, i.e. the 2nd integral in eq. (9).

$$I_2 = \int_{t_1}^{t_2} dt = -\int_{L_x+\delta L}^{0} \left[1 + \frac{\ddot{h}^+}{4c^2} x^2\right] \frac{dx}{c}$$

where $dt = -\left[1 + \frac{\ddot{h}^+}{4c^2} x^2\right]\frac{dx}{c}$

and ~~$x(t) = L_x\left[1 - \frac{c(t-t_1)}{L_x}\right]$~~

$x(t) = L_x - c(t-t_1)$ and

$x(t_2) = 0 \Rightarrow L_x = c(t_2 - t_1)$

$$I_2 = -\int_{L_x+\delta L}^{0} \frac{dx}{c} - \int_{L_x}^{0} \frac{\ddot{h}^+}{4c^2} x^2 \frac{dx}{c} + O(h^2) = \frac{L_x}{c} + \frac{L_x}{2c} h^+(t_1) + I_{2a}$$

$$I_{2a} = \int_{L_x}^{0} \frac{\ddot{h}^+}{4c^2} x^2 \left(-\frac{dx}{c}\right) = \int_{t_1}^{t_2} \frac{\ddot{h}^+}{4c^2} x^2 \, dt = \frac{\dot{h}^+}{4c^2} x^2 \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\dot{h}^+}{4c^2} 2x \cdot \dot{x} \, dt$$

in this case $\dot{x} = -c$.

$$I_{2a} = 0 - \frac{\dot{h}^+(t_1)}{4c^2} L_x^2 + \int_{t_1}^{t_2} \frac{\dot{h}^+}{2c} \cdot x \, dt = -\left(\frac{L_x}{2c}\right)^2 \dot{h}^+(t_1) + \frac{h^+}{2c} x \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \frac{h^+}{2} dt =$$

$$= -\left(\frac{L_x}{2c}\right)^2 \dot{h}^+(t_1) - \frac{h^+(t_1)}{2c} L_x + \frac{h_0}{2\Omega} \left[\sin\left[\Omega\left(t_1 + \frac{L_x}{c}\right)\right] - \sin(\Omega t_1)\right]$$

Putting all together, the second integral I_2 becomes:

$$I_2 = \frac{L_x}{c} - \left(\frac{L_x}{2c}\right)^2 \dot{h}^+(t_1) + \frac{h_0}{2\Omega} \left[\sin\left[\Omega\left(t_1 + \frac{L_x}{c}\right)\right] - \sin(\Omega t_1)\right] \quad (21)$$

The round trip time is given by (20) and (21):

$$t_2 - t_0 = I_1 + I_2 = \frac{2L_x}{c} + \frac{h_0}{2\Omega} \left\{\sin\left[\Omega\left(t_0 + \frac{2L_x}{c}\right)\right] - \sin(\Omega t_0)\right\}$$

Developing the parenthesis:

$$\{\ \} = \left\{\sin\left(\frac{2\Omega L_x}{c}\right)\cos(\Omega t_0) - \left[1 - \cos\left(\frac{2\Omega L_x}{c}\right)\right] \cdot \sin(\Omega t_0)\right\}$$

We can aproximate this expression because, for Virgo for exemple:

$$\frac{2\Omega L_x}{c} = \frac{4\pi f L_x}{c} = \frac{4\pi \cdot 10^2 \cdot 3 \cdot 10^3}{3 \cdot 10^8} \simeq 1.2\,\% \quad \text{so:}$$

$$\{\ \} \simeq \sin\left(\frac{2\Omega L_x}{c}\right) \cdot \cos(\Omega t_0)$$

The final result, considering the aproximation above, is:

$$t_2 - t_0 = \frac{2L_x}{c}\left\{1 + \frac{\sin\left(\frac{2\Omega L_x}{c}\right)}{\frac{2\Omega L_x}{c}} \cdot \frac{h_0}{2}\cos(\Omega t_0)\right\} \quad (22)$$

For a photon that propagates along the y-axis the same calculation yeolds :

$$t_2' - t_0 = \frac{2L_y}{c}\left\{1 - \frac{\sin\left(\frac{2\Omega L_y}{c}\right)}{\frac{2\Omega L_y}{c}} \frac{h_0}{2}\cos(\Omega t_0)\right\} \quad (23)$$

If we consider that $L_x = L_y + \Delta L$ where $\frac{\Delta L}{L_{x,y}} \ll 1$ then the phase difference between the 2 e.m. waves is :

$$\Delta\varphi = \pi + \omega(t_2 - t_0) - \omega(t_2' - t_0) =$$

$$= \underbrace{\pi + \frac{2\Delta L}{c}\omega}_{\text{time constant term}} + \frac{2L}{c}\cdot\omega \underbrace{\frac{\sin\left(\frac{2\Omega L}{c}\right)}{\frac{2\Omega L}{c}}}_{\text{typical Sinc response}} \cdot h_0 \cdot \cos(\Omega t_0) \quad (24)$$

the π comes from the working principle of the beamsplitter.
The intensity of light out of the interferometer is

$$I = \frac{I_0}{2}\left[1 - \cos(\varphi_0 + \delta\varphi)\right] \quad \text{where}$$

$$\varphi_0 = \pi + \frac{2\Delta L}{c}\cdot\omega$$

$$\delta\varphi = \operatorname{sinc}\left(\frac{2\Omega L}{c}\right)\cdot 2\frac{\omega}{c}L \cdot h_0 \cdot \cos(\Omega t_0)$$