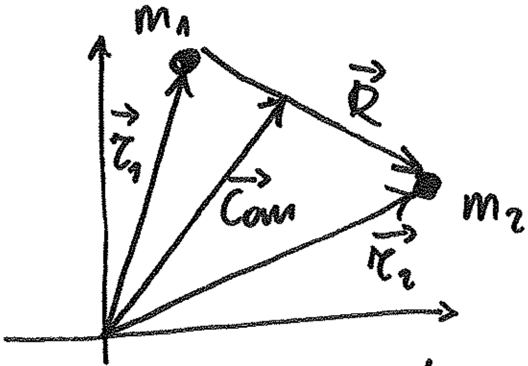
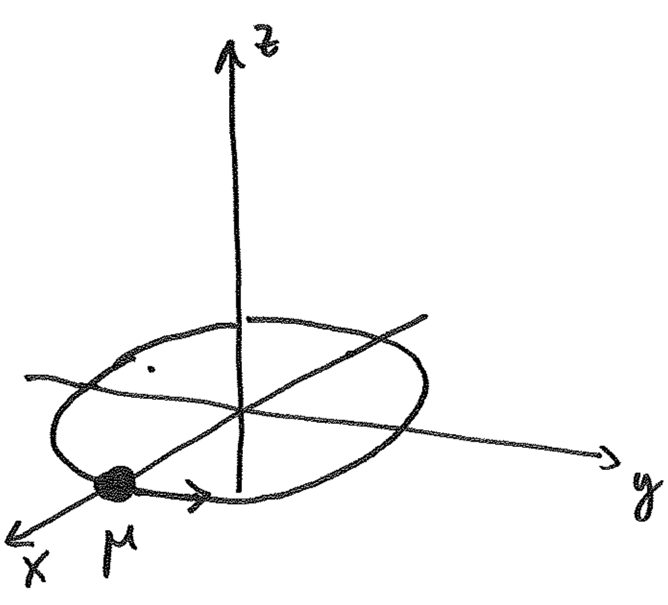
GW from a binary in a circular orbit

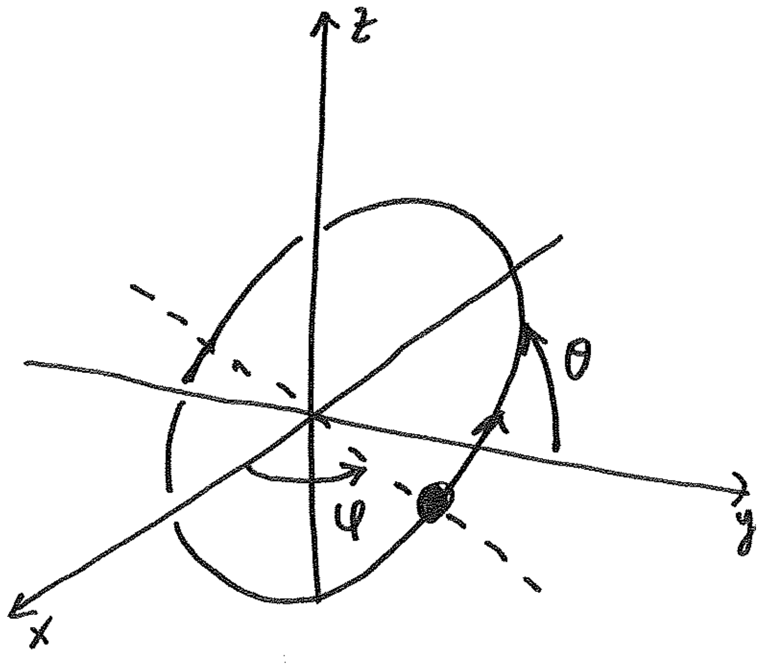
We have 2 point-like objects of masses m_1 and m_2. in a circular orbit around the centre of mass of the system.
The positions of the two masses are given by $\vec{r}_1(t)$ and $\vec{r}_2(t)$.
We change coordinates introducing the vector position of the centre of mass $\vec{Com}$ and relative distance $\vec{R}(t)$:

$$\vec{Com} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \qquad \vec{R} = \vec{r}_2 - \vec{r}_1$$



The system is isolated so, $\vec{Com}$ we can assume that is constant with time. The dynamic is reduced to a single point of mass $\mu = \frac{m_1 m_2}{m_1+m_2}$ orbiting at constant distance $|\vec{R}|$ around the origin of coordinates 0.
We consider the orbit laying on the x-y plane.
The GW propagates in the z direction, positive:





The kinematics of the particle on the plane is given by:

$$\vec{R} = R\begin{pmatrix}\cos(wt)\\ \sin(wt)\\ 0\end{pmatrix} = R\begin{pmatrix}c_w\\ s_w\\ 0\end{pmatrix}$$

we use a simplified notation of the trigonometric function sin and cos.

To describe a general circular orbit we have to introduce 2 rotation matrices, R_θ and R_φ:

$$R_\theta = \begin{pmatrix}1 & 0 & 0\\ 0 & c_\theta & -s_\theta\\ 0 & s_\theta & c_\theta\end{pmatrix} \qquad R_\varphi = \begin{pmatrix}c_\varphi & -s_\varphi & 0\\ s_\varphi & c_\varphi & 0\\ 0 & 0 & 1\end{pmatrix}$$

the tilted orbit kinematics is then:

$$\vec{R}' = R_\theta \cdot \vec{R} = R\begin{pmatrix}c_w\\ c_\theta \cdot s_w\\ s_\theta \cdot s_w\end{pmatrix} \qquad \vec{R}'' = R_\varphi \cdot \vec{R}' = R\begin{pmatrix}c_\varphi \cdot c_w - s_\varphi \cdot c_\theta \cdot s_w\\ s_\varphi \cdot c_w + c_\varphi \cdot c_\theta \cdot s_w\\ s_\theta \cdot s_w\end{pmatrix}$$

~

The quadrupole radiation expression is

$$h^{TT}_{ij}(t,\vec{x}) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}\,\ddot{\mathcal{M}}^{kl}$$

where:

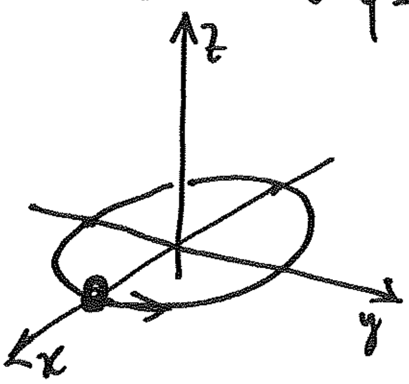
$$\mathcal{M}^{kl} = \frac{1}{c^2}\int d^3x'\, x^k \cdot x^l \cdot T^{00} = \sum_{i=1}^{N} x_i^k \cdot x_i^l \cdot m_i$$

If $\hat{n} = \hat{k}$ then:

$$\Lambda_{ij,kl}\,\ddot{\mathcal{M}}^{kl} = \begin{pmatrix}\frac{\ddot{\mathcal{M}}_{11} - \ddot{\mathcal{M}}_{22}}{2} & \ddot{\mathcal{M}}_{12} & 0\\ \ddot{\mathcal{M}}_{12} & -\frac{\ddot{\mathcal{M}}_{11} - \ddot{\mathcal{M}}_{22}}{2} & 0\\ 0 & 0 & 0\end{pmatrix}$$

Now we study different cases corresponding at different orientations of tho orbit.

Case 1 $\quad \theta = \varphi = 0 \quad \vec{r}'' = \vec{r} = R\begin{pmatrix} c_\omega \\ s_\omega \\ 0 \end{pmatrix} \quad \mathcal{M}_{11} = +\mu R^2 c_\omega^2 = +\mu\frac{R^2}{2}(1+c_{2\omega})$



$$\mathcal{M}_{12} = \mu R^2 \cdot c_\omega \cdot s_\omega = \mu\frac{R^2}{2}s_{2\omega} \qquad \mathcal{M}_{22} = +\mu\frac{R^2}{2}(1-c_{2\omega})$$

So

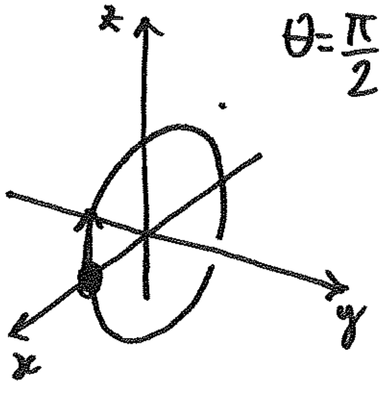
$$h^+ = -\frac{1}{r}\frac{2G}{c^4}\mu\frac{R^2}{4}4\omega^2 \cdot c_{2\omega}\cdot 2 = -\frac{1}{r}\frac{4G}{c^4}\mu R^2\omega^2 \cdot c_{2\omega}$$

$$h^\times = -\frac{1}{r}\frac{2G}{c^4}\mu\frac{R^2}{2}4\omega^2 \cdot s_{2\omega} = -\frac{1}{r}\frac{4G}{c^4}\mu R^2\omega^2 \cdot s_{2\omega}$$

The solution shows the 2 polarizations present with the same amplitude and out of phase $\frac{\pi}{2}$. It means the radiation is perfectly circularly polarized.

Case 2 $\quad \varphi = 0,\ \theta = \frac{\pi}{2} \quad \vec{r}'' = R\begin{pmatrix} c_\omega \\ 0 \\ s_\omega \end{pmatrix} \quad \mathcal{M}_{11} = \mu\frac{R^2}{2}(1+c_{2\omega})$

$$\mathcal{M}_{22} = 0 \quad \mathcal{M}_{12} = 0$$

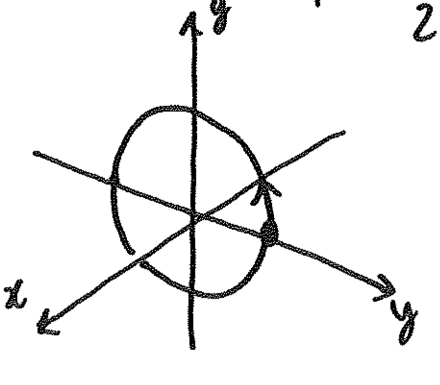


$$h^+ = -\frac{1}{r}\frac{2G}{c^4}\mu\frac{R^2}{4}4\omega^2 \cdot c_{2\omega} = -\frac{1}{r}\frac{2G}{c^4}\mu R^2\omega^2 \cdot c_{2\omega}$$

$$h^\times = 0$$

Case 3 $\quad \varphi = \theta = \frac{\pi}{2} \quad \vec{r}'' = R\begin{pmatrix} 0 \\ c_\omega \\ s_\omega \end{pmatrix} \quad \mathcal{M}_{11} = \mathcal{M}_{12} = 0$

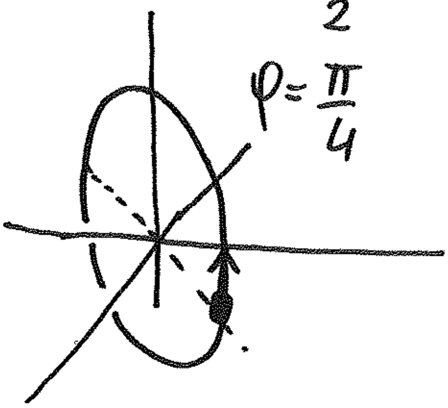
$$\mathcal{M}_{22} = \mu\frac{R^2}{2}(1+c_{2\omega})$$



$$h^+ = -\frac{1}{r}\frac{2G}{c^4}\mu\frac{R^2}{4}4\omega^2 \cdot c_{2\omega} = -\frac{1}{r}\frac{2G}{c^4}\mu R^2\omega^2 \cdot c_{2\omega}$$

$$h^\times = 0$$

Case 4 $\quad \theta = \frac{\pi}{2}, \quad \varphi = \frac{\pi}{4}$



$$\vec{r}'' = R \begin{pmatrix} \frac{c_\omega}{\sqrt{2}} \\ \frac{c_\omega}{\sqrt{2}} \\ s_\omega \end{pmatrix} \qquad \mathcal{M}_{11} = \mu \frac{R^2}{2}\frac{1}{2}(1+c_{2\omega}) = \mathcal{M}_{22} = \mathcal{M}_{12} \quad \text{So,}$$

$$h^+ = 0$$

$$h^\times = -\frac{1}{r}\frac{2G}{c^4}\mu\frac{R^2}{4}4\omega^2 \cdot c_{2\omega} = -\frac{1}{r}\frac{2G}{c^4}\mu R^2\omega^2 \cdot c_{2\omega}$$

Cases 3 and 4 should come from the rotation of the "detector" around the z axis by a suitable angle η.
The rotation matrix is:

$$R_\eta = \begin{pmatrix} c_\eta & s_\eta & 0 \\ -s_\eta & c_\eta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The tensor h_{ij} changes into $R^T \tilde{h} R$

$$\tilde{h} \cdot R = \begin{pmatrix} h^+ & h^\times & 0 \\ h^\times & -h^+ & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$R^T_\eta = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} h^+c - h^\times s & h^+s + h^\times c & 0 \\ h^+s + h^\times c & h^\times s - h^+c & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R^T \tilde{h} R = \begin{pmatrix} h^+c^2 - h^\times cs - h^+s^2 - h^\times cs & h^+cs + h^\times c^2 - h^\times s^2 + h^+cs & 0 \\ h^+cs - h^\times s^2 + h^+cs + h^\times c^2 & h^+s^2 + h^\times cs + h^\times cs - h^+c^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} h^+c_{2\eta} - h^\times s_{2\eta} & h^\times c_{2\eta} + h^+ s_{2\eta} & 0 \\ h^\times c_{2\eta} + h^+ s_{2\eta} & -h^+c_{2\eta} + h^\times s_{2\eta} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A rotation of $-45°$ produces a change of polarization from $h^\times$ to h^+.

Let's see a rotation around the x-axis.

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_x & s_x \\ 0 & -s_x & c_x \end{pmatrix} \qquad \tilde{h}\cdot R_x = \begin{pmatrix} h^+ & h^\times & 0 \\ h^\times & -h^+ & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_x & s_x \\ 0 & -s_x & c_x \end{pmatrix} = \begin{pmatrix} h^+ & h^\times c_x & h^\times s_x \\ h^\times & -h^+ c_x & -h^+ s_x \\ 0 & 0 & 0 \end{pmatrix}$$

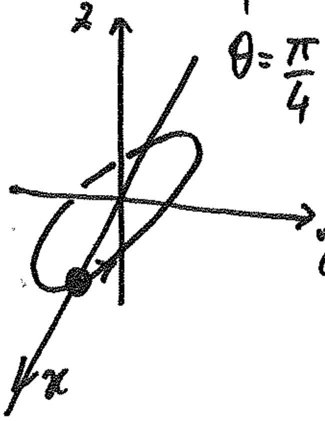
$$R_x^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_x & -s_x \\ 0 & s_x & c_x \end{pmatrix} \quad R_x^T \tilde{h} R_x = \begin{pmatrix} h^+ & h^\times c_x & h^\times s_x \\ h^\times c_x & -h^+ c_x^2 & -h^+ s_x c_x \\ h^\times s_x & -h^+ c_x s_x & -h^+ s_x^2 \end{pmatrix}$$

For $x = \frac{\pi}{2}$ one has: $\begin{pmatrix} h^+ & 0 & h^\times \\ 0 & 0 & 0 \\ h^\times & 0 & -h^+ \end{pmatrix}$

Case 5

$\varphi = 0$

$\theta = \frac{\pi}{4}$



$$\vec{r}'' = R \begin{pmatrix} c_\omega \\ \frac{s_\omega}{\sqrt{2}} \\ \frac{s_\omega}{\sqrt{2}} \end{pmatrix} \qquad \mathcal{H}_{11} = \mu \frac{R^2}{2}(1 + c_{2\omega})$$

$$\mathcal{H}_{22} = \mu \frac{R^2}{4}(1 - c_{2\omega})$$

$$\mathcal{H}_{12} = \mu \frac{R^2}{2\sqrt{2}} s_{2\omega}$$

$$h^+ = -\frac{1}{r}\frac{2G}{c^4}\mu\frac{R^2}{2}\frac{4\omega^2}{2}\cdot c_{2\omega}\cdot\left(1 + \frac{1}{2}\right) = -\frac{1}{r}\frac{3G}{2c^4}\mu R^2\omega^2\cdot c_{2\omega}$$

$$h^\times = -\frac{1}{r}\frac{2G}{c^4}\mu\frac{R^2}{2\sqrt{2}}4\omega^2\cdot s_{2\omega} = -\frac{1}{r}\frac{4G}{\sqrt{2}c^4}\mu R^2\omega^2\cdot s_{2\omega}$$

Case 6

$\varphi = 0$

$\theta\ \forall$

$$\vec{r}'' = R\begin{pmatrix} c_\omega \\ c_\theta s_\omega \\ s_\theta s_\omega \end{pmatrix} \qquad \mathcal{H}_{11} = \mu\frac{R^2}{2}(1 + c_{2\omega}) \quad \mathcal{H}_{22} = \mu\frac{R^2}{2}c_\theta^2(1 - c_{2\omega})$$

$$\mathcal{H}_{12} = \mu\frac{R^2}{2}c_\theta s_{2\omega}$$

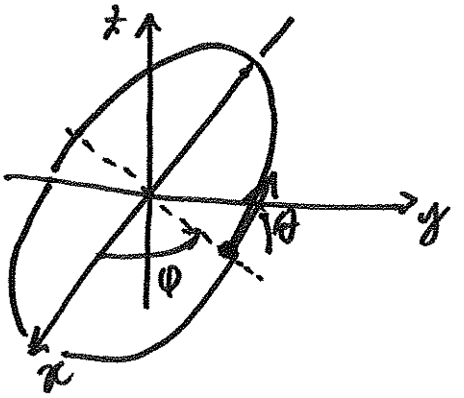
$$h^+ = -\frac{1}{r}\frac{2G}{c^4}\mu\frac{R^2}{2}4\omega^2\frac{1}{2}c_{2\omega}(1 + c_\theta^2) = -\frac{1}{r}\frac{4G}{c^4}\mu R^2\omega^2\left(\frac{1 + c_\theta^2}{2}\right)\cdot c_{2\omega}$$

$$h^\times = -\frac{1}{r}\frac{2G}{c^4}\mu\frac{R^2}{2}4\omega^2\cdot c_\theta s_{2\omega} = -\frac{1}{r}\frac{4G}{c^4}\mu R^2\omega^2\cdot c_\theta\cdot s_{2\omega}$$

A remarcable result of this case is the fact that the ratio between the amplitudes gives the inclination θ of the orbit

$$\frac{\sqrt{h^{+} h^{+*}}}{\sqrt{h^{\times} h^{\times *}}} = \frac{1+\cos^2\theta}{2\cos\theta}$$

Case 7 this is the general case



$$\vec{r}\,'' = R \begin{pmatrix} c_\varphi \cdot c_\omega - s_\varphi \cdot c_\theta \cdot s_\omega \\ s_\varphi \cdot c_\omega + c_\varphi \cdot c_\theta \cdot s_\omega \\ s_\theta \cdot s_\omega \end{pmatrix}$$

$$M_{11} = \mu \frac{R^2}{2}\left[c_\varphi^2 (1+c_{2\omega}) + s_\varphi^2 \cdot c_\theta^2 (1-c_{2\omega}) - s_{2\varphi} \cdot c_\theta \cdot s_{2\omega}\right]$$

$$M_{22} = \mu \frac{R^2}{2}\left[s_\varphi^2 (1+c_{2\omega}) + c_\varphi^2 \cdot c_\theta^2 (1-c_{2\omega}) + s_{2\varphi} \cdot c_\theta \cdot s_{2\omega}\right]$$

$$h^{+} = -\frac{1}{r}\frac{2G}{c^4}\mu\frac{R^2}{2}4\omega^2\frac{1}{2}\left[c_{2\omega}\left(c_\varphi^2 - s_\varphi^2 \cdot c_\theta^2 - s_\varphi^2 + c_\varphi^2 \cdot c_\theta^2\right) - s_{2\omega}\left(s_{2\varphi} \cdot c_\theta \cdot 2\right)\right] =$$

$$= -\frac{1}{r}\frac{2G}{c^4}\mu R^2 \omega^2\left[c_{2\omega}\left(c_{2\varphi} + c_\theta^2 \cdot c_{2\varphi}\right) - s_{2\omega}\left(2 \cdot s_{2\varphi} \cdot c_\theta\right)\right] =$$

$$= -\frac{1}{r}\frac{2G}{c^4}\mu R^2 \omega^2\left[(1+c_\theta^2) \cdot c_{2\varphi} \cdot c_{2\omega} - c_\theta \cdot s_{2\varphi} \cdot 2 \cdot s_{2\omega}\right]$$

$$M_{12} = \mu\frac{R^2}{2}\left[\frac{1}{2}s_{2\varphi}(1+c_{2\omega}) - \frac{1}{2}s_{2\varphi} \cdot c_\theta^2 \cdot (1-c_{2\omega}) + \left(c_\varphi^2 \cdot c_\theta - s_\varphi^2 \cdot c_\theta\right)s_{2\omega}\right]$$

$$h^{\times} = -\frac{1}{r}\frac{2G}{c^4}\mu\frac{R^2}{2}4\omega^2\left[c_{2\omega}\left(\frac{1}{2}s_{2\varphi} + \frac{1}{2}s_{2\varphi} \cdot c_\theta^2\right) + c_\theta \, c_{2\varphi} \cdot s_{2\omega}\right] =$$

$$= -\frac{1}{r}\frac{2G}{c^4}\mu R^2 \omega^2\left[(1+c_\theta^2) \cdot s_{2\varphi} \cdot c_{2\omega} + c_\theta \cdot c_{2\varphi} \cdot 2 s_{2\omega}\right]$$

$$h^+ = -\frac{1}{r}\frac{4G}{c^4}\mu R^2\omega^2\left[c_{2\varphi}\cdot\frac{1+c_\theta^2}{2}c_{2\omega} - s_{2\varphi}\cdot c_\theta\cdot s_{2\omega}\right]$$

$$h^\times = -\frac{1}{r}\frac{4G}{c^4}\mu R^2\omega^2\left[s_{2\varphi}\cdot\frac{1+c_\theta^2}{2}c_{2\omega} + c_{2\varphi}\cdot c_\theta\cdot s_{2\omega}\right]$$

which is case 5 rotated by an angle φ around the z-axis. Let's see if h^+ and $h^\times$ are still in quadrature. Multiplying the 2 [ ] we have:

$$\frac{s_{4\varphi}}{2}\cdot\left(\frac{1+c_\theta^2}{2}\right)^2 c_{2\omega}^2 - \frac{s_{4\varphi}}{2}\cdot c_\theta^2\cdot s_{2\omega}^2 + O(c_{2\omega}\cdot s_{2\omega})$$ the DC part is:

$$\frac{s_{4\varphi}}{2}\frac{1}{2}\left[\frac{(1+c_\theta^2)^2}{4} - c_\theta^2\right]$$ which is 0 only if $\theta = 0$.