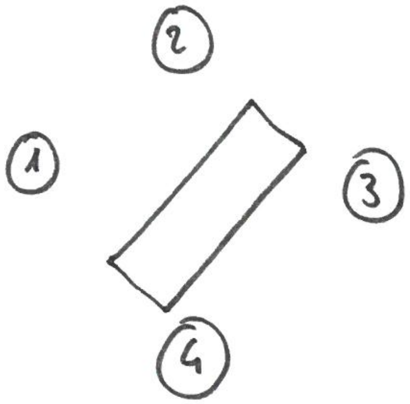
The beam splitter

$$\begin{pmatrix} 0 & r & t & 0 \\ r & 0 & 0 & t \\ t' & 0 & 0 & r' \\ 0 & t' & r' & 0 \end{pmatrix} \begin{pmatrix} e_1^i \\ e_2^i \\ e_3^i \\ e_4^i \end{pmatrix} = \begin{pmatrix} r e_2 + t e_3 \\ r e_1 + t e_4 \\ t' e_1 + r' e_4 \\ t' e_2 + r' e_3 \end{pmatrix}$$



$\boxed{S^{\dagger} S^{*} = \mathbb{1} \quad \text{ENERGY CONSERVATION}}$

Symmetric case :

$$\begin{pmatrix} 0 & r & t & 0 \\ r & 0 & 0 & t \\ t & 0 & 0 & r \\ 0 & t & r & 0 \end{pmatrix} \begin{pmatrix} 0 & r^* & t^* & 0 \\ r^* & 0 & 0 & t^* \\ t^* & 0 & 0 & r^* \\ 0 & t^* & r^* & 0 \end{pmatrix} =$$

$\Rightarrow$ ① $r r^* + t t^* = 1$ $\qquad r = |r| \cdot e^{i\varphi} \quad t = |t| e^{i\psi}$

② $r t^* + t r^* = 0$

① $\longrightarrow$ $|r|^2 + |t|^2 = 1$

② $\longrightarrow$ $e^{i(\varphi-\psi)} + e^{-i(-\psi+\varphi)} = 0 \Rightarrow \cos(\varphi-\psi) = 0$

$$\varphi = \psi + (2n+1)\frac{\pi}{2}$$

Real coefficients case

$$\begin{pmatrix} 0 & r & t' & 0 \\ r & 0 & 0 & t' \\ t & 0 & 0 & r' \\ 0 & t & r' & 0 \end{pmatrix} \begin{pmatrix} 0 & r^* & t^* & 0 \\ r^* & 0 & 0 & t^* \\ t^{*'} & 0 & 0 & r^{*'} \\ 0 & t^{*'} & r^{*'} & 0 \end{pmatrix}$$

$$|r|^2 + |t'|^2 = 1$$

$$|t|^2 + |r'|^2 = 1$$

$$r t^* + t' r^{*'} = 0$$

$r, t, r^*, t^* \in \mathbb{R}$

we choose $t = t'$ then

$$\Longrightarrow r = -r'$$

$$\frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$